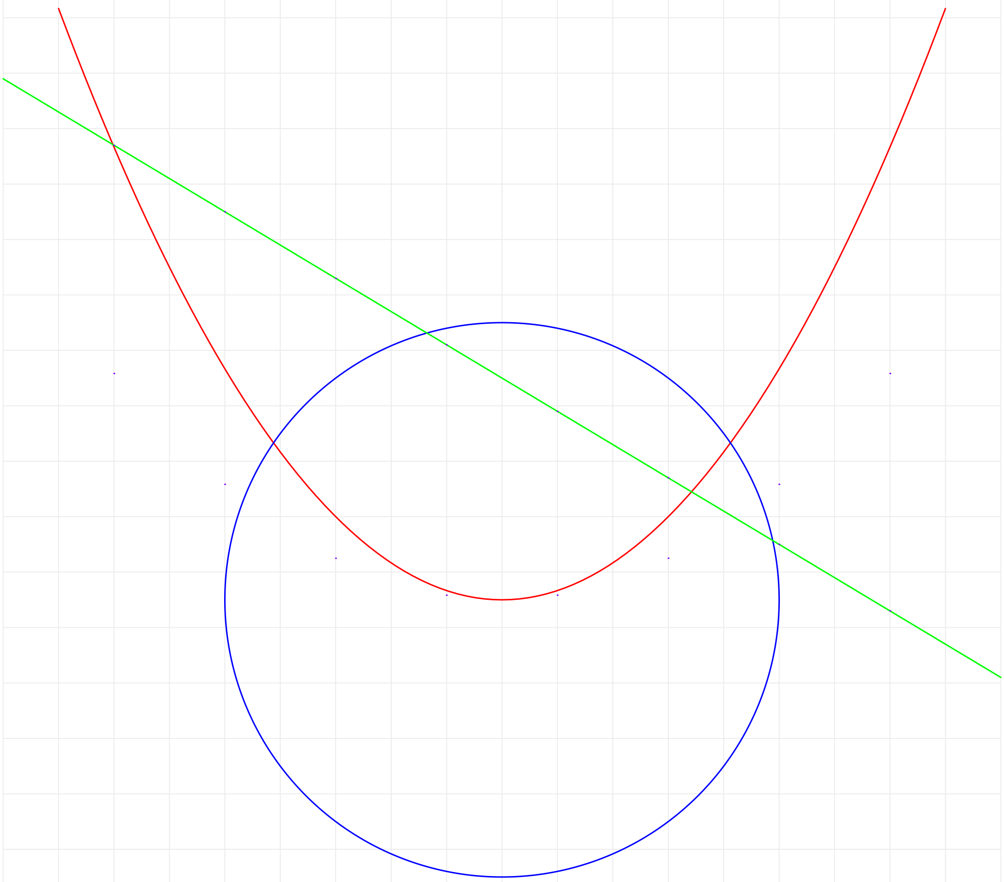
# A Point on Coordinate Geometry

A Problem Oriented Approach

Shiv Shankar Dayal



*Om Shivay Swaha!*

*Idam Shivay Idann Mam!!*

*(It is dedicated to Lord Shiv. It is his and not mine.)*

**A Point on Coordinate Geometry**

Early Draft[May 20, 2026]

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# Preface

This book provides an introduction to coordinate geomtry. In this book we cover striaght lines, pair of straight lines, circle, parabla, hyperbola, ellipse and general conics in two-dimensions. We also cover planes, lines, spheres, cones etc in the three diemnsional coordinate geometry.

## Prerequisite

Basic mathematics till grade 10 is needed to understand this book. Basic knowledge of trigonometry and geometry are needed to understand the material presented in this book.

## Acknowledgements

I am in great debt of my family and free software community because both of these groups have been integral part of my life. Family has provided direct support while free software community has provided the freedom and freed me from the slavery, which comes as a package with commercial software. I am especially grateful to my wife, children and parents because it is their time, which I have borrowed to put in the book. To pay my thanks from free software community I will take one name and that is Richard Stallman who started all this and is still fighting this never-ending war. He has been called the *hero* of the mankind by Donald Knuth for a reason.

This book is written using `Typst` developed by Laurenz Mädje and Martin Haug(thanks to both of them for such a nice project). I have used LaTeX and ConTeXt earlier but Typst is crazy fast for typesetting so I chose it for this book.

I have used Asymptote for drawing all the diagrams. Asymptote is very powerful and its syntax is close to mainstream programming languages unlike Metapost and Tikz. Not to mention the fact that it can do real 3D, though, this book will not make any usage of 3D features. Many thanks to Andy Hammerlindl, John C. Bowman, and Tom Prince for creating such a wonderful drawing application.

I would like to thank my parents, wife(Binita Rubi), and children for taking out their fair share of time and the support, which they have extended to me during my bad times. After that I would like to pay my most sincere gratitude to my teachers particularly H. N. Singh, Yogendra Yadav, Satyanand Satyarthi, Kumar Shailesh and Prof. T. K. Basu. Now is the turn of people from software community. I must thank the entire free software community for all the resources they have developed to make computing better. However, few names I know and here they go. Richard Stallman is the first, Donald Knuth, Edger Dijkstra, John von Neumann after that. I am not a native English speaker, and this book has just gone through one pair of eyes therefore chances are high that it will have lots of errors(particularly with commas and spelling mistakes). At the same time it may contain technical errors. With time and revisiosn those errors will be removed.

Shiv Shankar Dayal

Nalanda, India, 2026

# Theory and Problems

# 1 Coordinates

Coordinate geometry is study of geometric using algebra. It is also called **Analytical Geometry**. In geometry, we study about points, lines, triangles, quadrilaterals, circles, polygons etc. without the use of algebra but as said in coordinate geometry these geometrical figures are studied using variables and equations of algebra. In coordinate geometry we will come to know a term called Cartesian plane or cartesian coordinates which are named after René Des Cartes who first published his work on coordinate geometry in 1637. Pierre de Fermat also independently discovered, but he did not publish his discovery.

## 1.1 Number Line

We take a straight line and any point $O$ on it. This point $O$ is taken such that it divides the line in two equal parts. We take the part right of this point as positive part and the part left of it as negative part of this line. We represent the number 0 with the point $O$. Let us take another point $A$ such that it represents 1 on the number line. Now $OA$ represents a unit, and thus we can represent all natural numbers in terms of it. We can also represent all real numbers on this number line. Positive real numbers will lie on the right side and negative real numbers will lie on the left side of the mid-point, which is $O$. Because all real numbers can be represented on this line we call it *real line* or in general *number line*.

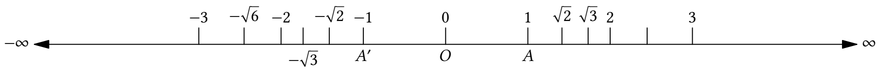



Figure 1.1: Number Line.

As we know from our basic algebra that there are a infinite number of rational and irrational numbers between any two numbers so we will have infinite points between any two points on this number line which we know from geometry.

## 1.2 Coordinates

Consider any point $P$ in a 2-dimensional plane. To define the location of this point we need a reference. In any n-dimensional plane no point can be given an absolute position rather we define location of points in relation to some other point in the same plane. Typically, we define axes of the plane and take intersection of these axes as origin, represented by $O$ and every other point is defined with reference to this point. This method gives us cartesian coordinates. The other way is choosing an origin and one axis. Every point is then defined in terms of its distance from origin and the angle made by the line joining the point and origin with the axis. This method gives us *Polar Coordiantes*. We will first study cartesian coorsinates, and later we will study polar coordinates.

### 1.2.1 Cartesian Coordinate System

In general, when we study cartesian coordinate system we mamke use of two perpendicular axes. However, to prevent the loss of generality let us make use of inclined axes or oblique axes. Consider the diagram given below:

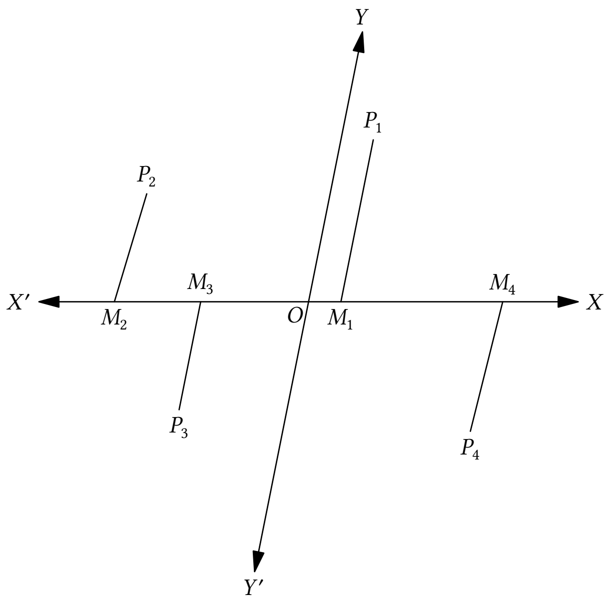



Figure 1.2: Cartesian axes in $xy$-plane.

$XOX'$ is called the $x$-axis and $YOY'$ is called the $y$-axis. As said the point $O$ is called the origin of coordinates or simply, the *origin*.

Consider a point $P_1$. Draw a line from it parallel to $OY$ to meet $OX$ at $M_1$. The distance $OM_1$ is called the *Abscissa* and the distance $M_1P$ is called the *Ordinate* of the point $P_1$, while together they are called its *Coordinate*.

Distances when measured parallel to $x$-axis are typically denoted by $x$ with or without a suffix, for example: $x_1, x_2, ..., x', x'', ...$, and distances when measured parallel to $y$-axis are typically denoted by $y$ with or without a suffix, for example: $y_1, y_2, ..., y', y'', ...$.

A point $P$ having the abscissa $x$ and the ordinate is $y$ is typically denoted as $(x, y)$. For example, if a point is at two units distance from $y$-axis and at a distance of three units from $x$-axis then it is written as $(2, 3)$.

Distances measured parallel to $OX$ are positive while distances measured parallel to $OX'$ are negative. Similarly, distances measured parallel to $OY$ are positive and distances parallel to $OY'$ are negative. Thus, cartesian plane is divided into four quadrants. $XOY$ is the first qudrant in which both abscissa and ordinates are positive. $X'OY$ represents the second quadrant in which abscissa is negative and ordinate is positive. $X'OY'$ is the third quadrant where both abscissa and ordinate are negative and finally $XOY'$ is the fourth quadrant in which abscissa is positive and ordinate is negative. Observe that these quadrants are taken in anti-clockwise order and similar to qudrants in trigonometry.

Referring to Figure 1.2 we can say that $P_1$ lies in first quadrant, $P_2$ lies in second quadrant and so on.

The axes, when they are not at right angles, are called *Oblique Axes*. The angle between positive $x$-axis and $y$-axis i.e. angle between $OX$ and $OY$ or the $\angle XOY$ is generally denoted by the Greek letter $\omega$.

In general, it is convenient to take the axes at right angles. The axes are then called the *Rectangular Axes*. You can assume that axes are rectangular unless otherwise stated. As you can guess this system is called Cartesian Coordinate System in the honor of Des Cartes.

## 1.3 Distance Between Two Points

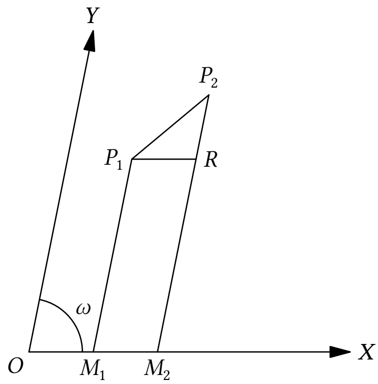



Figure 1.3: Distance between two points.

Let $P_1$ and $P_2$ be two points as shown in the diagram with coordiantes $(x_1, y_1)$ and $(x_2, y_2)$. $M_1P_1 \parallel OY, M_2P_2 \parallel OY$ and $P_2R \parallel OX$. Thus, $P_1R = M_2M_1 = OM_2 - OM_1 = x_2 - x_1$, $RP_2 = M_2P_2 - M_1P_1 = y_2 - y_1$ and $\angle P_1RP_2 = \angle OM_1P_1 = 180^\circ - P_2M_2X = 180^\circ - \omega$.

From trigonometry, we know that $P_1P_2^2 = P_2R^2 + RP_1^2 - 2P_2R.RP_1\cos(180^\circ - \omega)$ which is

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega \tag{1.1}$$

In the case when axes are rectangular i.e. axes are at right angle or $\omega = 90^\circ$ then the Equation (1.1) reduces to $P_1P_2^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ and therefore in rectangular coodinate system the distance between two points is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Clearly, the distance between any point $(x, y)$ and the origin $(0, 0)$ will be $\sqrt{x^2 + y^2}$

## 1.4 Section Formula

Consider the diagram given below:

Consider points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. The point $P(x, y)$ divides the line segment $P_1P_2$ such that $P_1P : PP_2 :: m_1; m_2$ then we need to find the coordinates of $P$.

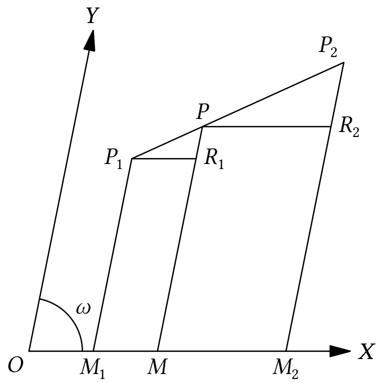



Figure 1.4: Section Formula.

We have $OM_1 = x_1, M_1P_1 = y_1, OM = x, MP = y, OM_2 = x_2$ and $M_2P_2 = y_2$.

$P_1R_1, PR_2 \parallel OX$. Then

$P_1R_1 = MM_1 = OM - OM_1 = x - x_1$

$PR_2 = MM_2 = OM_2 - OM = x_2 - x$

$R_1P = MP - M_1P_1 = y - y_1$

and $R_2P_2 = M_2P_2 - MP = y_2 - y$.

From the similar triangles $\triangle P_1R_1P$ and $\triangle PP_2R_2$, we have

$\frac{m_1}{m_2} = \frac{P_1P}{PP_2} = \frac{P_1R_1}{PR_2} = \frac{x-x_1}{x_2-x}$

$\therefore x = \frac{m_1x_2+m_2x_1}{m_1+m_2}$. Similarly, it can be found that $y = \frac{m_1y_2+m_2y_1}{m_1+m_2}$

Thus, the coordinates of the point which divides $P_1P_2$ in the ratio $m_1 : m_2$ is

$$(x, y) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}\right) \tag{1.2}$$

If the point $P$ divides these two points externally then we can prove in a similar manner that the point's coordinates is

$$(x, y) = \left(\frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \frac{m_1y_2 - m_2y_1}{m_1 - m_2}\right) \tag{1.3}$$

Clearly if the point is mid-point then we find that the coordinates of mid-point is given by

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \tag{1.4}$$

## 1.5 Area of a Trapezium

We will consider the area of a trapezium in this section from geometry because it will be useful in the coming sections. Consider the following diagram:

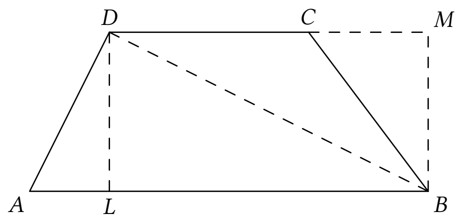



Figure 1.5: Area of a trapezium.

We know that area of a triangle is half of the product of height of any side and the perpendicular drawn from the opposite angle. Thus,

Area $ABCD = \Delta ABC + \Delta ACD = \frac{1}{2}.BC.AL + \frac{1}{2}.AD.CN = \frac{1}{2}(BC + AD).AL$

## 1.6 Area of a Triangle

Consider following diagram with $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$:

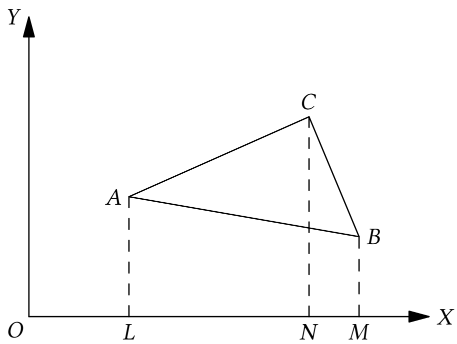



Figure 1.6: Area of a triangle.

Let $\Delta$ denote the area of the $\triangle\, ABC$. Then, $\Delta ABC = \square ALNC + \square CNMB - \square ALMB$

$= \frac{1}{2}LN(LA + NC) + \frac{1}{2}NM(BC + MB) - \frac{1}{2}LM(LA + MB)$

$= \frac{1}{2}[(x_3 - x_1)(y_1 + y_3) + (x_2 - x_3)(y_2 + y_3) - (x_2 - x_1)(y_1 + y_2)]$

On simplification, we obtain

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \tag{1.5}$$

The equation can be represented in the determinant form as given below:

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{1.6}$$

In the above calculation if the axes are oblique with an angle $\omega$ then the area of the triangle would be:

$$\Delta = \frac{1}{2}\sin\omega[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \tag{1.7}$$

If one of the vertices is the origin $O(0, 0)$ then the area of the triangle would be $\frac{1}{2}(x_1y_2 - x_2y_1)$.

### 1.6.1 Sign of Area of a Triangle

As you can see from the formula; area of a triangle could be positive or negative. There are two ways to fix this. The first is to take modulus of the equation. The second is to plot the points of triangle in clockwise direction. If we plot the points in anti-clockwise direction then area of the triangle will be negative in that case we can take modulus of the value obtained.

### 1.6.2 Condition of Collinearity of Three Points

From the above relation we can extrapolate that if the area of the triangle is zero then the three vertexes would collapse into a straight line. Thus the condition of collinearity of these three points can be written as:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \tag{1.8}$$

Now that we have area of a trapezium and area of a triangle finding area of other polygons is trivial and we will find them in our exercises.

## 1.7 Centroid of a Triangle

The point of intersection of the medians of the triangle is called the *centroid* of the triangle.

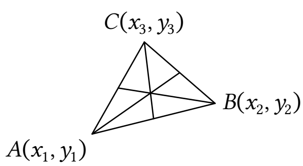



Figure 1.7: Centroid of a triangle.

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the $\triangle\, ABC$. Let $AD, BE$ and $CF$ be the three medians.

Since $D$ is middle point of $BC \Rightarrow D = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Let $G$ be the centroid i.e. intersection of the three medians. $G$ will divide $BC$ in the ratio $2:1$ i.e. $AG : GC = 2 : 1$ then

$G = \left(\frac{2\frac{x_2+x_3}{2}+1.x_1}{2}, \frac{2\frac{y_2+y_3}{2}+1.y_1}{2}\right) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{2}\right)$

Similarly it can be shown that $G$ has same coordinates for other medians. Thus, $G$ lies on the same coordinates for all three medians.

Thus, $G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{2}\right)$ is the centroid of the triangle.

## 1.8 Incenter of a Triangle

The point of intersection of the bisectors of the angles of the triangle is called the *incenter* of the triangle.

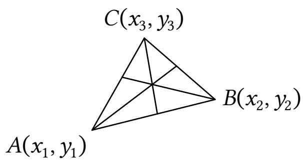



Figure 1.8: Incenter of a triangle.

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the $\triangle\ ABC$. Let $AD, BE$ and $CF$ be the three internal bisectors of the angles $A, B$ and $C$, respectively. Let these bisectors meet at the incenter $I$.

Since $BC$ is internal bisector of $\angle BAC$, therefore $\frac{BD}{DC} = \frac{BA}{AC} = \frac{c}{b}$

$\Rightarrow D = \left(\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}\right)$

The incenter $I$ divides $AD$ internally in the ratio $b + c : a = .AI : ID = b + c : a$

$\Rightarrow I = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

Similarly it can be shown for two other bisectors that $I$ has the same coordinate.

Thus,

$$I = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right) \tag{1.9}$$

is the incenter of the triangle.

## 1.9 Area of a Quadrilateral

Consider following diagram with $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ and $D(x_4, y_4)$:

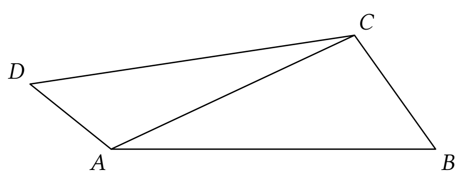



Figure 1.9: Area of a quadrilateral.

If $A, B, C, D$ are in anti-clockwise order then area of quadrilateral will be positiive but if it is clockwise then it will be negative and we will have to take modulus of it.

Area of $\square ABCD = \Delta ABC + \Delta ACD$

Another way to find area of a quadrilateral is

$$\square ABCD = \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 y_2 & \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & y_4 \\ x_1 & y_1 \end{vmatrix}\right] \tag{1.10}$$

## 1.10 Polar Coordinates

Consider a line $OX$ through a point $O$. We call the line *initial line* and we call the *the pole* or *the origin.*

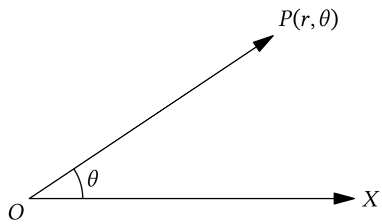



Figure 1.10: Polar Coordinates.

As you can figure out the position of a point $P$ can be found by knowing its distance from the pole and the positive counter-clockwise angle made by the line joining the point with the initial line. The distance $OP$ is typically denoted by $r$ and the $\angle XOP$ is denoted by $\theta$. $OP$ or $r$ is called the *radius vector* and $\theta$ is called the *vectorial angle* of the point.

## 1.11 Polar Coordinates and Cartesian Coordinates

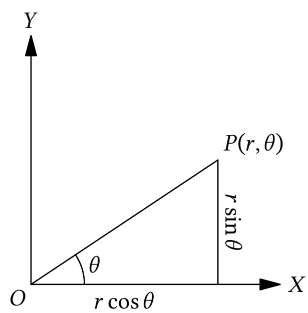



Figure 1.11: Polar Cartesian Coordinates.

As we can see from the diagram a point $P$ having polar coordinates $(r, \theta)$ can be represented in terms of cartesian coordinate as $(r\cos\theta, r\sin\theta)$. Similarly, if a point has cartesian coordinates $(x, y)$ then $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} \frac{y}{x}$ would represent the equivalent polar coordinates.

## 1.12 Problems

Find the areas of the triangles the coordinates of whose vertices are respectively:

1. $(1, 3), (-7, 6)$ and $(5, -1)$.
2. $(0, 4), (3, 6)$ and $(-8, -2)$.
3. $(5, 2), (-9, -3)$ and $(-3, -5)$.
4. $(a, b + c), (a, b - c)$ and $(-a, c)$.
5. $(a \cos \varphi_1, a \sin \varphi_1), (a \cos \varphi_2, a \sin \varphi_2)$ and $(a \cos \varphi_3, a \sin \varphi_3)$.
6. $(am_1^2, 2am_1), (am_2^2, 2am_2)$ and $(am_3^2, 2am_3)$.

Prove by showing that the area of the triangle formed by them is zero that the following points are in a straight line:

7. $(1, 4), (3, -2)$ and $(-3, 16)$.
8. $\left(-\frac{1}{2}, 3\right), (-5, 6)$ and $(-8, 8)$.
9. $(a, b + c), (b, c + a)$ and $(c, a + b)$.
10. In any $\triangle ABC$ prove that $AB^2 + AC^2 = 2(AC^2 + CD^2)$ where $D$ is the middle point of $BC$.
11. $ABC$ is a triangle and $D, E$ and $F$ are the middle points of the sides $BC, CA$ and $AB$; prove that the point which divides $AD$ internally in the ratio $2 : 1$ also divides the line $BE$ and $CF$ in the same ratio.
12. Find the area of a quadrilateral whose vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and $(x_4, y_4)$.

Find the areas of the quadrilaterals the coordinates of whose vertices, taken in order, are

13. $(1, 1), (3, 4), (5, -2)$ and $(4, -7)$.
14. $(-1, 0), (-3, -9), (5, 8)$ and $(3, 9)$.

Find the lengths of the straight lines joining the pairs of points whose polar coordinates are

15. $(r_1, \theta_1)$ and $(r_2, \theta_2)$.
16. $(2, 30^\circ)$ and $(4, 120^\circ)$.
17. $(-3, 45^\circ)$ and $(7, 105^\circ)$.
18. $\left(a, \frac{\pi}{2}\right)$ and $\left(3a, \frac{\pi}{6}\right)$.

Find the areas of the triangle whose coordinates are

19. $(r_1, \theta_1), (r_2, \theta_2)$ and $(r_3, \theta_3)$.
20. $(1, 30^\circ), (2, 60^\circ)$ and $(3, 90^\circ)$.

21. $(-3, 30^\circ), (5, 150^\circ)$ and $(7, 210^\circ)$.
22. $\left(-a, \frac{\pi}{6}\right), \left(a, \frac{\pi}{2}\right)$ and $\left(-2a, \frac{2\pi}{3}\right)$.

Find the distance between the points.

23. $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$.
24. $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

Change the following equations to polar coordinates

25. $x^2 + y^2 = a^2$.
26. $y = x\tan\alpha$.
27. $x^2 + y^2 = 2ax$.
28. $x^2 - y^2 = 2ay$.
29. $x^2 = y^2(2a - x)$.
30. $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

Change the following equations to cartesian coordinates

31. $r = a$.
32. $\theta = \tan^{-1} m$.
33. $r = a\cos\theta$.
34. $r^2 = a^2\sin 2\theta$.
35. $r^2\sin 2\theta = 2a^2$.
36. $\sqrt{r}\cos\frac{\theta}{2} = \sqrt{a}$.
37. $\sqrt{r} = \sqrt{a}\sin\frac{\theta}{2}$.
38. $r(\cos 3\theta + \sin 3\theta) = 5k\sin\theta\cos\theta$.
39. Find $a$ if the distance between $(a, 2)$ and $(3, 4)$ is 8.
40. Prove that the distance between the points $(a + r\cos\theta, b + r\sin\theta)$ and $(a, b)$ is independent of $\theta$.
41. Use distance formula to show that the points $(\csc^2\theta, 0), (0, \sec^2\theta)$ and $(1, 1)$ are collinear.
42. If the point $P(x, y)$ be equidistant from the points $(a + b, b - a)$ and $(a - b, a + b)$, prove that $\frac{a-b}{a+b} = \frac{x-y}{x+y}$.
43. Prove that the points $(3, 4), (8, -6)$ and $(13, 9)$are the vertices of a right angled triangle.
44. Determine the type(isosceles, right angled, right angled isosceles, equilateral, scalene) of the following triangles whose vertices are
    1. $(1, 1), \left(-\sqrt{3}, \sqrt{3}\right)$ and $(-1, -1)$.
    2. $(0, 2), (7, 0)$ and $(2, 5)$.

3. $(-2, 5), (7, 10)$ and $(3, -4)$.

45. Prove that the distance of the point $(a\cos\alpha, a\sin\alpha)$ from the origin is independent of $\alpha$.
46. Let $A(6, -1), B(1, 3)$ and $C(x, 8)$ be three points such that $AB = BC$. Find the value of $x$.
47. Using distance formula, show that the points $(1, 5), (2, 4)$ and $(3, 3)$ are collinear.
48. Prove that the points $(2a, 4a), (2a, 6a)$ and $2a + \sqrt{3a}, 5a$ are the vertices of an equilateral triangle.
49. If the line segment joining the points $A(a, b)$ and $A(a, b)$ subtend an angle $\theta$ at the origin $O$, prove that $\cos\theta = \frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$ or $OA.OB.\cos\theta = ac + bd$.
50. Find the circumcenter of the triangle whose vertices are $(-2, -3), (-1, 0)$ and $(7, -6)$. Also find the radius of the circumcircle.
51. The distance between two parallel lines is unity. A point $P$ lies between the lines at a distance $a$ from one of them. Find the length of a side of an equilateral $\triangle$ $PQR$, vertex $Q$ of which lies on one of the parallel lines and vertex $R$ on the other line.
52. The opposite angular points of a square are $(3, 4)$ and $(1, -1)$, find the coordinate of the remaining vertices of the square.
53. $A(-4, 0)$ and $B(-1, 4)$ are two given points. $C$ and $D$ are symmetric to the given points $A$ and $B$ respectively with respect to $y$-axis. Calculate the perimeter of the trapezium $ABCD$.
54. Point $B$ is symmetric to $A(4, -1)$ with respect to the bisector of the first quadrant. Find $AB$.
55. A line segment $AB$ through a point $A(2, 0)$ which makes an angle of $30^{\circ}$ with the positive direction of $x$-axis is rotated about $A$ in anti-clockwise direction though an angle of $15^{\circ}$. If $C$ be the new position of the point $B\left(2 + \sqrt{3}, 1\right)$, find the coordiantes of $C$.
56. The point $(1, -2)$ is reflected in the $x$-axis and then translated parallel to the positive direction of $x$-axis through a distance of 3 units. Find the coordinates of the point in the new position.
57. The line segment joining $A(3, 0)$ and $B(5, 2)$ is rotated about $A$ in the anti-clockwise direction by an angle of $45^{\circ}$ so that point $B$ goes to $C$. If $D$ is the reflection of $C$ in $y$-axis, find the coordinates of $D$.
58. Find the coordinates of the point which divides the line segment joining the points $(5, -2)$ and $(9, 6)$ internally and externally in the ratio $3 : 1$.
59. $x$ coordinates of two points $B$ and $C$ are the roots of the equation $x^2 + 4x + 3 = 0$ and their $y$ coordinates are the roots of the equation $x^2 - x - 6 = 0$. If $x$ coordinates of $B$ is less than $x$ coordinate of $C$ and $y$ coordinate of $B$ is greater than the $y$ coordinate of $C$ and coordinate of a third point $A$ be $(3, -5)$, find the length of the bisector of the interior angle at $A$.

60. Find the ratio in which the point $(2,y)$ divides the line segment joining $(4,3)$ and $(6,3)$ and hence find the value of $y$.

61. If $A,B,C,D$ are points whose coordinates are $(-2,3),(8,9),(0,4)$ and $(3,0)$ respectively, find the ratio in which $AB$ is divided by $CD$.

62. If $A_1,A_2,A_3,...,A_n$ are $n$ points in a plane whose coordinates are $(x_1,y_1),(x_2,y_2),(x_3,y_3),...,(x_n,y_n)$ respectively. $A_1A_2$ is bisected in the point $G_1$; $G_1A_3$ is divided at $G_2$ in the ratio $1:2$; $G_2A_4$ is divided at $G_3$ in the ratio $1:3$; and so on until all the points are exhausted. Show that the coordinates of the final point so obtained are
$$\frac{x_1+x_2+...+x_n}{n}, \frac{y_1+y_2+...+y_n}{n} \tag{1.11}$$
.

63. Show that the straight line $ax+by+c=0$ divides the join of points $A(x_1,y_1)$ and $B(x_2,y_2)$ in the ratio $-\frac{ax_1+by_1+c}{ax_2+by_2+c}$. Explain the negative sign.

64. A line $L$ intersects three sides $BC,CA$ and $AB$ of a $\triangle\, ABC$ in $P,Q$ and $R$ respectively. Show that $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=-1$.

65. The vertices of a triangle are $A(x_1,x_1\tan\alpha),B(x_2,x_2\tan\beta),C(x_3,x_3\tan\gamma)$. If the circumcenter of $\triangle\, ABC$ coincides with the origin and $H(\overline{x},\overline{y})$ is the orthocenter of $\triangle\, ABC$, (where $x_1\sec\alpha,x_2\sec\beta,x_3\sec\gamma$ are of the same sign). Show that $\frac{\overline{y}}{\overline{x}}=\frac{\sin\alpha+\sin\beta+\sin\gamma}{\cos\alpha+\cos\beta+\cos\gamma}$.

66. If $\alpha,\beta$ and γ are the real roots of the equation $x^3-3px^2+3qx-1=0$, find the centroid of the triangle whose vertices are $\left(\alpha,\frac{1}{\alpha}\right),\left(\beta,\frac{1}{\beta}\right)$ and $\left(\gamma,\frac{1}{\gamma}\right)$.

67. If $A(at^2,2at),B\left(\frac{a}{t^2},-\frac{2a}{t}\right)$ and $C(a,0)$ be any three points, show that $\frac{1}{AC}+\frac{1}{BC}$ is independent of $t$.

68. If two vertices of an equilateral triangle be $(0,0)$ and $\left(3,\sqrt{3}\right)$, find the coordinate of the third vertex.

69. Find the circumcenter and circumradius of the triangle whose vertices are $(-2,3),(2,-1)$ and $(4,0)$.

70. The vertices of a triangle are $A(1,1),B(4,5)$ and $C(6,13)$. Find $\cos A$.

71. Find the distance between the points $\left(3,\frac{\pi}{4}\right)$ and $\left(7,\frac{5\pi}{4}\right)$.

72. $A(2,4)$ and $B(2,6)$ are two given points; $ABP$ is an equilateral triangle on the side of $AB$ opposite to the origin. Find the coordinates of $P$.

73. Show that the points $(2,45^\circ),\left(\sqrt{2},90^\circ\right)$ and $(-2,135^\circ)$ are the vertices of a right angled triangle.

74. Find the coordinates of the point which divides the line segment joining $(2,4)$ and $(6,8)$ in the ratio $1:3$ internally and externally.

75. Find the coordinates of the points which trisect the line segment joining the points

1. $(1,-2)$ and $(-3,4)$.
2. $(2,3)$ and (6, 5).

76. $A(1,1)$ and $B(2,-3)$ are two points and $D$ is a point on $AB$ produced such that $AD = 3AB$, then find the coordinates of $D$.
77. If the middle point of the line segment joining $(3,4)$ and $(k,7)$ is $(x,y)$ and $2x+2y+1=0$, find the value of $k$.
78. One end of a diameter of a circle is at $(2,3)$ and the center is $(-2,5)$, find the coordinates of the other end of the diameter.
79. Find the length of the medians of the triangle whose vertices are $(-1,3),(1,-1)$ and $(5,1)$.
80. If the point $C(-1,2)$ divides the line segment joining $A(2,5)$ and $B$in the ratio $3:4$, find the coordinates of $B$.
81. $A,B,C$ are collinear points and $B$ lies between $A$ and $C$. $A$ and $B$ are $(3,4)$ and $(7,7)$ respectively. If $AC = 10$ units, find the coordinates of $C$.
82. Find the ratio in which $(-8,3)$ divides the line segment of the points $(2,-2)$ and $(-4,1)$.
83. In what ratio does the $x$-axis divides the line segment joining the points $(2,-3)$ and $(5,6)$.
84. Show that the straight lines joining the points $A(0,-1)$ and $B(15,2)$ divides the line joining the points $C(-1,2)$ and $D(4,-5)$ internally in the ratio $2:3$.
85. Find the ratio in which the line segment joining the points $(1,2)$ and $(-2,3)$ is divided by the line $3x+4y=7$.
86. Find the ratio in which the line $y-x+2=0$ divides the line segment joining $(3,-1)$ and $(8,9)$.
87. Find the distance of the point from origin which divides the line segment joining the points $(5,-4)$ and $(3,-2)$ in the ratio $4:3$.
88. The coordinates of the middle points of the sides of a triangle are $(1,1),(2,3)$ and $(4,1)$, find the coordinates of the vertices.
89. Find the centroid and incenter of the triangle whose vertices are
    1. $(2,4),(6,4)$ and $(2,0)$.
    2. $(1,2),(2,3)$ and $(3,4)$.
90. Two vertices of a triangle are $(-1,4)$ and $(5,2)$. If its centroid is $(0,-3)$, find the third vertex.
91. $A(1,4)$ and $B(4,8)$ are two points. $P$ is a point on $AB$ such that $AP = AB + BP$. If $AP = 10$, find the coordinates of $P$.
92. Find the area of the triangle whose vertices $A,B,C$ are respectively $(3,4),(-4,3)$ and $(8,6)$.

93. Find the area of the quadrilateral whose vertices are $(-3,2),(7,-6),(-5,-4)$ and $(5,4)$.

94. The coordinates of points $A,B,C$ and $P$ are $(6,3),(-3,5),(4,-2)$ and (x, y) respectively, prove that $\frac{\Delta PBC}{\Delta ABC} = \frac{|x+y-2|}{7}$.

95. Show that the points (3, 3), (h, 0) and $(0,k)$ are collinear if $\frac{1}{h}+\frac{1}{k}=\frac{1}{3}$.

96. If $A(x_1,y_1),B(x_2,y_2)$ and $C(x_3,y_3)$ are the vertices of a $\triangle\, ABC$ and $(x,y)$ be a point on the internal bisector of $\angle A$, then prove that $b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$, where $AC=b$ and $AB=c$.

97. If the points $\left(\frac{a^3}{a-1},\frac{a^2-3}{a-1}\right),\left(\frac{b^3}{b-1},\frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1},\frac{c^2-3}{c-1}\right)$ are collinear for three distinct values $a,b$ and $c$ then show that $abc-(ab+bc+ca)+3(a+b+c)=0$.

98. If $(1,4)$ be the center of gravity of a triangle and the coordinates of its any two vertices be $(4,-8)$ and $(-9,7)$, find the area of the triangle.

99. Prove that the coordinates of the vertices of an equilateral triangle cannot all be rational.

100. If $A,B,C$ are the points $(-1,5),(3,1),(5,7)$ respectively and $D,E,F$ are the middle points of $BC,CA$ and $AB$ respectively prove that $\Delta ABC = 4\Delta DEF$.

101. The vertices of a $\triangle\, ABC$ are $A(3,0),B(0,6)$ and $C(6,9)$. A straight line $DE$ divides $AB$ and $AC$ in the ratio $1:2$ at $D$ and $E$ respectively, prove that $\Delta ABC = 9\Delta ADE$.

102. If $(t,t-2),(t+3,t)$ and $(t+2,t+2)$ are the vertices of a triangle, show that its area if independent of $t$.

103. If $A(x,y),B(1,2)$ and $C(2,1)$ are the vertices of a triangle of area 6 units, show that $x+y=15$ or $-9$.

104. Find the area of the quadrilateral whose vertices are $(1,1),(7,3),(12,2)$ and $(7,21)$.

105. Find the area of the pentagon whose vertices are $(4,3),(-5,6),(0,-7),(3,-6)$ and $(-7,-2)$.

106. Find the area of the hexagon whose consecutive vertices are $(5,0),(4,2),(1,3),(-2,2),(-3,-1)$ and $(0,-4)$.

107. Find the area of the triangle whose vertices are $((a+1)(a+2),a+2),((a+2)(a+3),a+3)$ and $((a+3)(a+4),a+4)$.

108. The point $A$ divides the join of $P(-5,1)$ and $Q(3,5)$ in the ratio $k:1$. Find the two values of $k$ for which the area of $\triangle\, ABC$, where $B(1,5)$ and $C(7,-2)$ are given, is equal to 2 units in magnitude.

109. The coordinates of $A,B,C,D$ are $(6,3),(-3,5),(4,-2),(x,3x)$ respectively. If $\Delta ABC = 2\Delta BCD$, find $x$.

110. If the area of the quadrilateral whose angular points taken in order are $(1, 2), (-5, 6), (7, -4)$ and (h,-2) be zero; show that $h = 3$.

111. Find the area of the triangle whose vertices $A, B, C$ are $(3, 4), (-4, 3), (8, 6)$ respectively and hence find the length of perpendicular from $A$ on $BC$.

112. The coordinates of the centroid of a triangle and those of two of its vertices are respectively $\left(\frac{2}{3}, 2\right), (2, 3), (-1, 2)$. Find the area of the triangle.

113. The area of a triangle is 3 square units. Two of its vertices are $A(3, 1), B(1, -3)$ and the centroid off the triangle lies on $x$-axis. Find the coordinates of the third vertex is $C$.

114. $A$ and $B$ are the points $(3, 4)$ and $(5, -2)$, find the point $P$ such that $PA = PB$ and $\Delta PAB = 10$.

115. Prove that the points $(a, b + c), (b, c + a)$ and $(c, a + b)$ are collinear.

116. If the points $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ be collinear, show that $\frac{y_2 - y_3}{x_2 x_3} + \frac{y_3 - y_1}{x_3 x_1} + \frac{y_1 - y_2}{x_1 x_2} = 0$.

117. If the points $(a, b), (a_1, b_1)$ and $(a - a_1, b - b_1)$ are collinear show that $\frac{a}{a_1} = \frac{b}{b_1}$.

118. Show that the points $(a, 0), (0, b)$ and $(1, 1)$ are collinear if $\frac{1}{a} + \frac{1}{b} = 1$.

119. Prove that the points $(-4, -1), (-2, -4), (4, 0)$ and $(2, 3)$ are the vertices of a rectangle.

120. If the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be the three consecutive vertices of a parallelogram, find the coordinates of the fourth vertex.

121. In any $\triangle\, ABC$, prove that $AB^2 + AC^2 = 2(AC^2 + BD^2)$, where $D$ is the middle point of $BC$.

122. If $G$ be the centroid of the $ABC$ and $O$ be any other point in the plane of the $ABC$, then prove that $OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$.

123. Prove that the area of a triangle is four times the area of the triangle formed by joining the mid points of its sides.

124. Prove that the line segment joining the middle points of two sides of a triangle is half the third side.

125. If $P, Q, R$ divide the sides $BC, CA$ and $AB$ of $\triangle\, ABC$ in the same ratio, prove that the centroid of both the triangles coincide.

126. Prove that in any triangle four times the sum of the squares of the medians is equal to three times the sum of the squares of the sides.

127. If $G$ be the centroid of a $\triangle\, ABC$, prove that $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$.

128. Show that the line joining the centroid of a triangle to its vertices divide it into three triangles of equal area.

129. Show that the middle point of the hypotenuse of a right angled triangle is equidistant from its vertices.

130. $A(a_1, b_1), B(a_2, b_2), C(a_3, b_3)$ are the vertices of a $\triangle\, ABC$. The side $AB$ is divided by the point $D$ in the ratio $\lambda : \mu$ and then the line segment $DC$ is divided by the point $E$ in the ratio $\mu : \lambda + \mu$. Find the coordinates of $E$.

131. The four points $A(\alpha, 0), B(\beta, 0), C(\gamma, 0)$ and $D(\delta, 0)$ are such that $\alpha, \beta$ are the roots of the equation $ax^2 + 2hx + b = 0$ and $\gamma, \delta$ are those of the equation $ax^2 + 2h'x + b' = 0$. Show that the sum of the ratios in which $C$ and $D$ divide $AB$ is zero if $ab' + a'b = 2hh'$.

132. $A(1, -2)$ and $B(2, 5)$ are two points. The lines $OA, OB$ are produced to $C$ and $D$ respectively such that $OC = 2OA$ and $OD = 2OB$. Find $CD$.

133. Two vertices of a triangle are $A(2, 1)$ and $B(3, -1)$. The third vertex $C$ lies on the line $y = x + 9$. If the centroid of $\triangle\, ABC$ lies on the $y$-axis, find the coordinates of $C$ and the centroid.

134. If $(\alpha, \beta), (x, y)$ and $(p, q)$ are the coordinates of the circumcenter, the centroid and the orthocenter of a triangle, prove that $3x = 2\alpha + p$ and $3y = 2\beta + p$.

135. Find the coordinates of the centroid, circumcenter and orthocenter of the triangle whose vertices are $(2, 3), (3, 4)$ and $(6, 8)$.

136. If $A\left(\alpha, \frac{1}{\alpha}\right), B\left(\beta, \frac{1}{\beta}\right)$ and $C\left(\gamma, \frac{1}{\gamma}\right)$ be the vertices of a $\triangle\, ABC$, where $\alpha.\beta$ are the roots of the equation $x^2 - 6p_1x + 2 = 0$; $\beta, \gamma$ are the roots of the equation $x^2 - 6p_2x + 3 = 0$ and $\gamma, \alpha$ are the roots of the equation $x^2 - 6p_x + 6 = 0$, $p_1, p_2, p_3$ being positive, find $p_1, p_2, p_3$ and the coordinates ofo the centroid of $\triangle\, ABC$.

137. If $\tan\alpha, \tan\beta, \tan\gamma$ are the roots of the equation $x^3 - 3ax^2 + 3bx - 1 = 0$, find the centroid of the triangle whose vertices are $(\tan\alpha, \cot\alpha), (\tan\beta, \cot\beta)$ and $(\tan\gamma, \cot\gamma)$.

138. Two unlike forces equal to 30 and 40 newtons are applied at the point $A(-3, -1)$ and $B(4, 6)$ respectively. Find the point of application of resultant force.

139. The area of a parallelogram is 12 units. Two of its vertices are the points $A(-1, 3)$ and $B(-2, 4)$. Find the other two vertices of the parallelogram if the point of intersection of diagonals lies on the positive side of $x$-axis.

140. Give the points $A(1, 2), B(8, 4), C(4, 10)$ find the coordinates of the point $P$ such that the triangles $PCB, PCA$ and $PAB$ have the same area in magnitude and sign.

141. If $a, b, c$ be the $p$th, $q$th and $r$th terms respectively of an H.P., show that the points $(bc, p), (ca, q), (ab, r)$ are collinear.

142. If $x_1, x_2, x_3$ are in A.P. and $y_1, y_2, y_3$ are also in A.P. prove that the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear.

143. If $a, b, c$ are distinct real numbers, show that the points $(a, a^2), (b, b^2)$ and $(c, c^2)$ are not collinear.

144. If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ are the vertices of a $\triangle\, ABC$ and $(x, y)$ be a point on the median through $A$, show that $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

145. The area of a triangle is $\frac{3}{2}$ sq. units. Two of its vertices are $A(2, -3)$ and $B(3, -2)$, the centroid of the triangle lies on the line $3x - y - 8 = 0$. Find the third vertex $C$.

146. Prove that the quadrilateral whose vertices are $A(-2, 5), B(4, -1), C(9, 1)$ and $D(3, 7)$ is a parallelogram and find its area. If $E$ divides $AC$ in the ratio $2 : 1$, prove that $D, E$ and the middle point $F$ of $BC$ are collinear.

147. Prove that points $(-3, -1), (2, -1), (1, 1)$ and $(-2, 1)$ taken in order are vertices of a trapezium.

148. If the vertices of a triangle have integral coordinates, prove that the triangle cannot be equilateral.

# 2 Locus

When the movement of a point is represented by an equation or equations then it means that the movement of point satisfies some geometric condition or conditions, the associated equation or equations or the path traced by the point is called its *locus*. In coordinate geometry all the equations whether they are straight lines, circles, parabolas, ellipses or hyperbolas are all locus of a point to some equation.

A common example is that of a circle. Consider a point $P$ which moves as to form a circle. Then the condition is that the distance of point from the center of the circle is equal to the radius of the circle. Let $C(a, b)$ be the center of the circle and $r$ be its radius. Then the equation which governs this movement is given by $(x-a)^2 + (y-b)^2 = r^2$, also known as general equation of a circle. In this case the circumference is the locus of the point $P$.

Another example could be given as bisector of a line segment. Consider a point $P$ and a line segment formed by two points $A$ and $B$. In this case the path is governed by the fact that $P$ moves in such a way that its distance from $A$ and $B$ remains equal. Let $A$ be $(x_1, y_1)$ and $B$ be $(x_2, y_2)$. Then we can say that $P(x, y)$ satisfies

$(x-x_1)^2 + (y-y_1^2) = (x-x_2^2) + (y-y_2)^2 \Rightarrow -2xx_1 + x_1^2 - 2yy_1 + y_1^2 = -2xx_2 + x_2^2 - 2yy_2 + y_2^2$

$\Rightarrow 2x(x_2 - x_1) + 2y(y_2 - y_1) = x_2^2 - x_1^2 + y_2^2 - y_1^2$ is the equation of the straight line which representd the locus of the bisector of $AB$, which is a straight line.

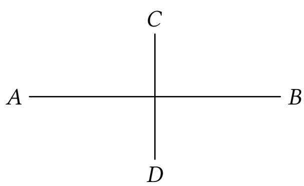



Figure 2.1: Bisector lines.

In the given figure both $AB$ and $CD$ are bisectors of each other.

We can also state our circle example in another manner, which will lead to same locus but a different equation. By geometry, we know that the end points of a diameter and any point on the circumference subtend a right angle at the circumference. Thus, $A$ and $B$ can be two different fixed points and $P$ can be a point such that $\triangle ABP$ is a right-angled triangle at $P$. The locus of $P$ is therefore again a circle with $AB$ as a diameter.

Consider equation of a straight line.

$$x = y \tag{2.1}$$

The equation Equation (2.1) is a relationship between two quantities and as such there are infinite solutions that exist. We cannot know all solutions but we can definitely enumerate some of the solutions. For example, $(x, y) \in \{(0,0), (1,1), (2,2), (3,3), (-1,-1), (-2,-2), (-3,-3)\}$ are some of the values which will satisfy the above equation. These are the points which will lie on the line represented by the equations Equation (2.1).

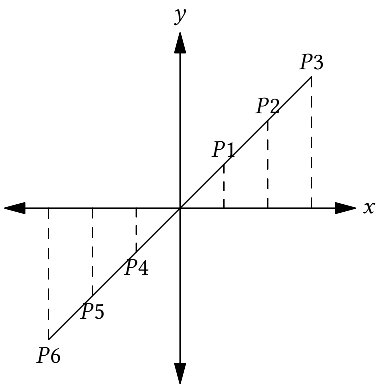



Figure 2.2: Locus of a straight line.

We see from Figure 2.2 that the points we have chosen indeed lie on the equation representing the locus.

Now we consider our example of a circle again. Imagine a circle with center at origin having radius of 2 units. The equation of this circle is given by

$$x^2 + y^2 = 1 \tag{2.2}$$

Like straight line there are infinite points on this circle and we consider some of them. For example, $(x, y) \in \{(2,0), \left(\sqrt{3},1\right), \left(\sqrt{2},\sqrt{2}\right), \left(1,\sqrt{3}\right), (0,2), \left(-1,\sqrt{3}\right), \left(-\sqrt{2},\sqrt{2}\right), \left(-\sqrt{3},1\right), (-2,0),$ $\left(-\sqrt{3},-1\right), \left(-\sqrt{2},-\sqrt{2}\right), \left(-1,-\sqrt{3}\right), (0,-2), \left(1,-\sqrt{3}\right), \left(\sqrt{2},-\sqrt{2}\right), \left(\sqrt{3},-1\right)\}$. These points are shown on the circle in the figure below:

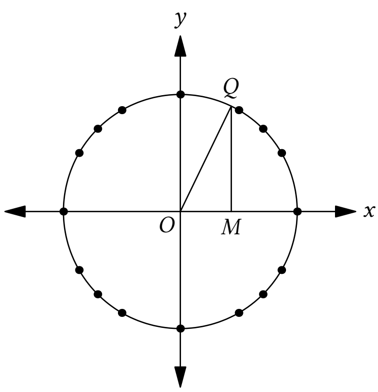



Figure 2.3: Locus of a circle.

We can take any point $Q(x, y)$ on Figure 2.3 such that $QM$ is the perpendicular on $x$-axis then we find that $QM^2 + OM^2 = OQ^2 \Rightarrow y^2 + x^2 = r^2$, which is what the equation of a circle is. Thus, all the points on the circle satisfy the Equation (2.2).

We take one more example of that of a parabola. We take a specific parabola given by the equation

$$y^2 = 4x \tag{2.3}$$

Like the circle example there are infinite points on this parabola as well and we consider some of them like the circle example. For example, $(x, y) \in \{(0, 0), (1, \pm 2),$ $\left(2, \pm 2\sqrt{2}, (4, \pm 4)\right\}$. These points are shown on the parabola in the figure below:

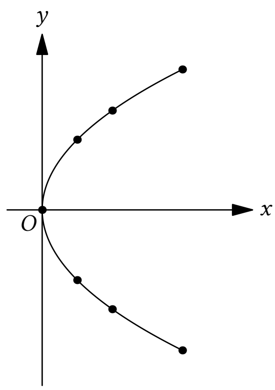



Figure 2.4: Locus of a parabola.

If we take any other point on the parabola then it will satisfy the equation of the parabola like it did for the circle example. Thus, all the points on the parabola satisfy the Equation (2.3).

Similarly, we can prove that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is locus of an ellipse and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is that of a hyperbola. You are encourage to draw the diagram and verify it.

## 2.1 Finding Locus of a Point

Let the coordinates of the moving point $P$ be $(\alpha, \beta)$. Apply the given geometrical constraint or condition tp obtain the relationship between $\alpha, \beta$ and known quantities. Replace $(\alpha, \beta)$ by $(x, y)$ in the obtained equation. The resulting equation will be the locus of the point under consideration.

If a point moves in such a manner that it satisfies any given condition it will describe some definitive curve, or locus then we can always find an equation between $x$ and $y$ of any point on the path. This equatio is called equation of the locus or the curve in question.

**Definition: Euqation to a curve:** *The equation to a curve is a relation, which exists between the coordinates of any point on the curve, and which holds for no other points except lying on the curve.*

It is obvious that for every equation between $x$ and $y$ a geometrical locus can always be found.

## 2.2 Equation of Locus in Different Forms

1. **Cartesian Form:** If the equation of the locus has the form $y = f(x)$ or $f(x, y) = 0$, where $x$ and $y$ are the cartesian coordinates of the point $P$, it is called cartesian equation of the locus. All the examples we have covered are of the cartesian form.
2. **Polar Form:** If the equation of the locus has the form $r = f(\theta)$ or $f(r, \theta) = 0$ are the polar coordinates of the point $P$, it is called polar equation of the locus. For example, $r^2 \cos^\theta + r^2 \sin^2\theta = a^2$ is the equation of the circle in polar form.
3. **Parametric Form:** If $x$ and $y$ coodinates of the point $P$ are given in terms of a third variable $t$(called the parameter). This equation is called parametric equation of the locus. For example, the equation of parabola, which is $y^2 = 4x$ in cartesian form can be written in parametric form as $x = t^2$ and $y = 2t$. If we eliminate $t$ then we get cartesian form of the equation to the locus.

## 2.3 Problems

1. A point moves so that the algebraic sum of its distances from two given perpendicular axes is equal to a constant quantity $a$; find the equation to its locus.
2. The sum of squares of the distances of a moving point from the two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quatity $2c^2$. Find the equation to its locus.
3. Find the locus of a point which moves such that its distance from the point $(-1, 0)$ is always three times its distance from the point $(0, 2)$.

$A$ and $B$ being the fixed points $(a, 0)$ and $(-a, 0)$ respectively, obtain the equations giving the locus of $P$, when

4. $PA^2 - PB^2 =$ a constant quantity $= 2k^2$.
5. $PA = nPB, n$ being a constant.
6. $PA + PB = c$, a constant quantity.
7. $PB^2 + PC^2 = 2PA^2, C$ being the point $(c, 0)$.
8. Find the locus of a point whose distance from the point $(1, 2)$ is equal to its distance from the $y$-axis.

Find the equation to the locus of a point which is always equidistant from the points whose coordinates are

9. $(1, 0)$ and $(0, -2)$.
10. $(2, 3)$ and $(4, 5)$.
11. $(a + b, a - b)$ and $(a - b, a + b)$.

Find the equation to the locus of a point which moves so that

12. its distance from the $x$-axis is three times its distance from the $y$-axis.

13. its distance from the point $(a, 0)$ is always four times its distance from the axis of $y$.
14. the sum of the squares of its distances from the axes is equal to 3.
15. the square of its distance from the point $(0, 2)$ is equal to 4.
16. its distance from the point $(3, 0)$ is three times its distance from $(0, 2)$.
17. its distance from the $x$-axis is always one half its distance from the origin.
18. A fixed point is at a perpendicular distance $a$ from a fixed straight line and a point moves so that its distance from the fixed point is always equal to its distance from the fixed line. Find the equation to its locus, the axes of coordinates being drawn through the fixed point and being parallel and perpendicular to the given line.
19. In the previous question if the first distance be (1), always half, and (2), always twice, the second distance, find the equations to the respective loci.
20. If the coordinates of a variable point $P$ be $(a\cos\theta, b\sin\theta)$, where $\theta$ is a variable quantity, find the locus of $P$.
21. Find the equation of the locus of a point such that the sum of its distances from $(0, 2)$ and $(0, -2)$ is 6.
22. $A$ and $B$ are two fixed points having a distance of $2a$. Find the locus of a point $P$ such that $\angle APB$ is a right angle.
23. A line segment $AB$ of length $a + b$ moves such that its extremeties $A$ and $B$ always remain on the axis of $x$ and $y$ respectively. Find the locus of a variable point $P$ on $AB$ such that $PA = a, PB = b$.
24. If $O$ be the origin and $A$ be a point on the locus $y^2 = 8x$. Find the locus of the middle point of $OA$.
25. Examine whether point $(2, -5)$ lies on the curve $x^2 + y^2 - 2x + 1 = 0$.
26. If the equations $ax^2 + 2hxy + by^2 = 0$ and $y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0$ represent the same curve, find $m_1 + m_2$ and $m_1 m_2$.
27. Find the locus of a variable point $(at^2, 2at)$, where $t$ is the parameter.
28. If the coordinates of a variable point $P$ be $\left(t + \frac{1}{t}, t - \frac{1}{t}\right)$, where $t$ is variable quantity, then find the locus of $P$.
29. If the coordinates of a variable point $P$ be $(\cos\theta + \sin\theta, \cos\theta - \sin\theta)$, where $\theta$ is a variable quantity, then find the locus of $P$.
30. If $A(\cos\theta, \sin\theta)$, $B(\sin\theta, \cos\theta)$ and $C(1, 2)$ are the vertices of a $\triangle\, ABC$, find the locus of its centroid if $\theta$ varies.
31. The position of a moving point in the $xy$-plane at time $t$ is $(u\cos\alpha.t, u\sin\alpha.t - kt^2)$, where $u, \alpha, k$ are constants. Find the locus of the moving point.
32. A point moves so that its distance from the point $(-2, 3)$ is always three times its distance from the point $(0, 3)$. Find equation to its locus.

33. $A$ and $B$ are two given points whose coordinates are $(-5, 3)$ and $(2, 4)$ respectively. A point $P$ moves in such a manner that $PA : PB = 3 : 2$. Find the equation to the locus of $P$.

34. $S$ is the point $(4, 0)$ and $M$ is the foot of perpendicular drawn from a point $P$ to the $y$-axis. If $P$ moves such that the distance $PS$ amd $PM$ remain equal, find the locus to $P$.

35. Prove that the locus of the point equidistant from two given points is the straight line which bisects the line segment joining the given points at right angles.

36. If a point $P$ moves such that its distance from $(a, 0)$ is always equal to $a + x$ coordinate of $P$, show that the locus of $P$ is $y^2 = 4ax$.

37. If $A(1, 2)$ and $B(-2, 3)$ are two fixed points, find the locus of a point $P$ so that area of $\triangle PAB$ is 9 units.

38. If $P$ is the middle point of the straight line joining a given point $A(1, 2)$ and $Q$ where $Q$ is a variable point on the curve $x^2 + y^2 + x + y = 0$. Find the locus of $P$.

39. $A(2, 3)$ is a fixed point and $Q(3\cos\theta, 2\sin\theta)$ a variable point. If $P$ divides $AQ$ internally in the ratio 3 : 1, find the locus of $P$.

40. From the point $A(6, -8)$ all possible lines are drawn to cut the $x$-axis. Find the locus of their middle points.

# 3 Straight Lines

Straight lines are the simplest locus which a point can have. We have studied the concept of locus in previous chapter. In this chapter we will study different forms of the straight lines and problems related to these concepts. It goes without saying that it is one of most fundamental and important concepts in coordinate geometry.

We know certain facts about straight lines:

1. Infinitely many lines can be drawn through a point. See Figure 3.1.

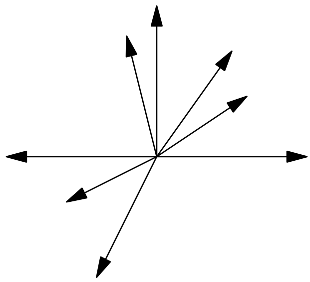


Figure 3.1:

2. Infinitely many lines can be drawn in a direction. See Figure 3.2.

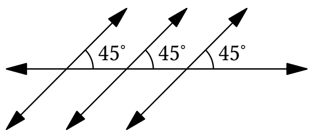



Figure 3.2:

3. One and only one line can be drawn through a fixed point in a given direction. See Figure 3.3.

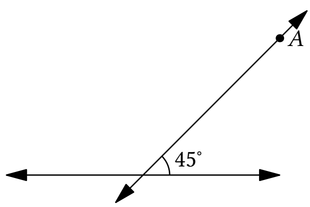



Figure 3.3:

4. One and only one line can be drawn through two given points. See Figure 3.4.

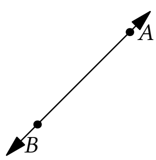



Figure 3.4:

# 3.1 Angle of Inclination of a Line

The angle of elevation of any line which cuts the $x$-axis the angle which it makes with $x$-axis in the positive anti-clockwise direction. See Figure 3.5.

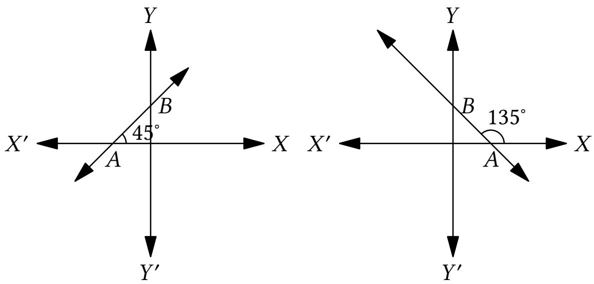



Figure 3.5:

# 3.2 Slope or Gradient of a Line

If the angle of inclination is $\theta$ then $\tan\theta$ is called the slope or gradient of the straight line. Usually it is denoted by $m$. Thus, the slope of the lines shown in Figure 3.5 are $\tan 45^{\circ}$ and $\tan 135^{\circ}$ i.e. 1 and $-1$.

If a line passes through two points $(x_1, y_1)$ and $(x_2, y_2)$ then the slope is given by $\frac{y_1 - y_2}{x_1 - x_2}$ as we will see soon.

# 3.3 Equation of a Straight Line

Equation of a straight line is the equation in $x$ and $y$, which is satisfied by the coordinates of all the points on the line and is not satisfied by any point which is not on the line. It is always an equation of first degree in $x$ and $y$. Thus, the general form would be $ax + by + c = 0$, where $a, b$ and $c$ are constants. There are other forms or representations of this equation which will follow this section. The equation may be an equation in $x$ and $y$ or only $x$ or only $y$.

# 3.4 Equation of Lines Parallel to Axes

## 3.4.1 Equation of $x$-axis

Let $P(x, y)$ be an arbitrary point on $x$-axis. Since $P(x, y)$ lies on $x$-axis, therefore, $y = 0$. The relation $y = 0$ is true for all points on $x$-axis, while $x$ varies. Hence the equation of $x$-axis is

$$y = 0. \tag{3.1}$$

### 3.4.2 Equation of $y$-axis

Let $P(x, y)$ be an arbitrary point on $y$-axis. Since $P(x, y)$ lies on $y$-axis, therefore, $x = 0$. The relation $x = 0$ is true for all points on $y$-axis, while $y$ varies. Hence the equation of $y$-axis is

$$x = 0. \tag{3.2}$$

### 3.4.3 Equations of Lines Parallel to $y$-axis

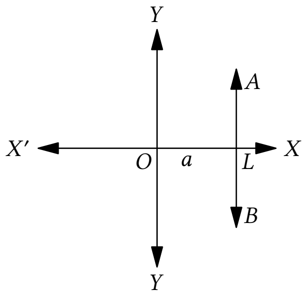



Figure 3.6:

Let $AB$ be a straight line parallel to $y$-axis at a distance $a$ from it on the positive side of $x$-axis.

Let $AB$ meets $x$-axis at $L$, then $OL = a$. Let $P(x, y)$ be any point on the line $AB$. Now $x = OL$ or $x = a$. This relation is satisfied for all points on the line $AB$ and is not satisfied by any point, which does not lie on $AB$.

Hence, the equation of the straight line parallel to $y$-axis at a distance $a$ from it on the positive side of $x$-axis is

$$x = a. \tag{3.3}$$

Similarly if the line is on negative side of $x$-axis is

$$x = -a. \tag{3.4}$$

### 3.4.4 Equations of Lines Parallel to $y$-axis

Proceeding like previous section, the equation of the straight line parallel to $x$-axis at a distance $b$ frfom it on the positive side of $y$-axis is

$$y = b. \tag{3.5}$$

and if on negative side is

$$y = -b. \tag{3.6}$$

## 3.5 Forms of Straight Line

### 3.5.1 Slope-Intercept Form

We will find an equation to a straight line whose slope is $m$ and which cuts an intercept on the $y$-axis i.e. which passes through the point $(0, c)$.

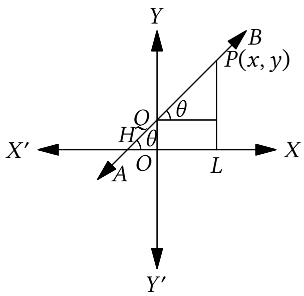



Figure 3.7: Slope intercept form of a straight line

Let $AB$ be a line whose slope is $m$ and which cuts an intercept $c$ on $y$-axis. Let $\angle BHX = \theta$ and line $AB$ cuts $y$-axis at $Q$.

Then from the figure $OQ = c$ and $\tan\theta = m$.

Let $P(x, y)$ be any point on the line $AB$. From $P$ we draw $PL \perp QX$ and from $Q$ we draw $QM \perp PL$.

Since $\angle PHL = \theta \therefore PQM = \theta$

Now in $\triangle PQM, \tan\theta = \frac{PM}{QM} = \frac{PL-ML}{Q}M = \frac{PL-QO}{O}L = \frac{y-c}{x} = m$

$$y = mx + c. \tag{3.7}$$

**Note:** If the line passes through origin, then $c = 0$ and the equation of the line becomes $y = mx$.

### 3.5.2 Point-Slope Form

We will find an equation to a straight line whose slope is $m$ and which passes through a point $(x_1, y_1)$. See Figure 3.8.

Let $AB$ be a straight line whose slope is $m$ and which passes through the point $Q(x_1, y_1)$. Let the line $AB$ cut the $x$-axis at $H$ and $\angle BHX = \theta$ then $\tan\theta = m$.

Let $P(x, y)$ be any point on line $AB$. From $P$ and $Q$ we draw $PL$ and $QM$ perpendiculars on $OX$ and from $Q$ we draw $QN$ perpedicular on $PL$.

Since $P \equiv (x, y) \therefore OL = x$ and $PL = y$

and since $Q \equiv (x_1, y_1) \therefore OM = x_1$ and $QM = y_1$

Since $\angle BHX = \theta \therefore \angle PQN = \theta$

Now from $\triangle PQN, \tan\theta = \frac{PN}{QN} = \frac{y-y_1}{x-x_1}$

$\therefore m = \frac{y-y_1}{x-x_1}$ or

$$y - y_1 = m(x - x_1). \tag{3.8}$$

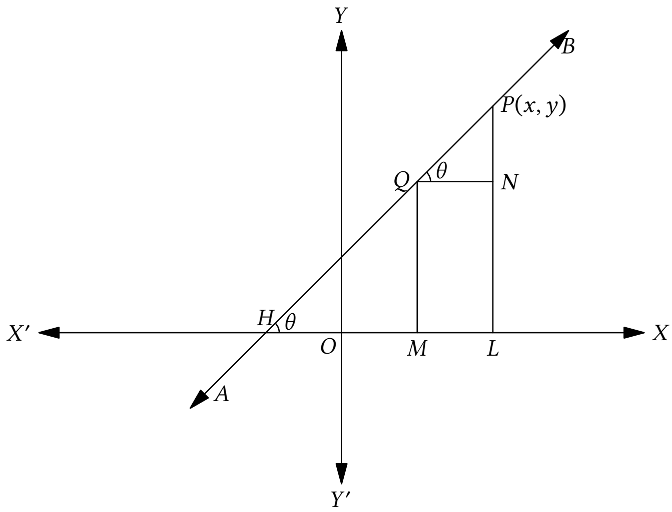



Figure 3.8: Slope point form of a straight line

**Aliter:** Since $m$ is the slope of the line therefore its equation may be written as $y = mx + c$. Since $(x_1, y_1)$ lies on this line therefore we can write

$y_1 = mx_1 + c \Rightarrow c = y_1 - mx_1$

Thus, $y - y_1 = m(x - x_1)$.

### 3.5.3 Two-Point Form

We will find an equation of a straight line which passes through two points $(x_1, y_1)$ and $(x_2, y_2)$. See Figure 3.9.

Let $AB$ be a line which passes through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$. Let $P(x, y)$ be any point on $AB$.

From $P, Q$ and $R$ we drop $PL, QM$ and $RN$ perpediculars to $x$-axis. From $Q$ we drop perpendicular $QH$ to $PL$ and from $R$ we drop perpendicular $RK$ to $QM$.

$\because P \equiv (x, y) \therefore OL = x, PL = y, \quad Q \equiv (x_1, y_1) \therefore OM = x_1, QM = y_1, \quad$ and $\quad \because R \equiv= (x_2, y_2) \therefore ON = x_2, RN = y_2$

From similar $\triangle\, PHQ$ and $\triangle\, QKR$, we have

$\frac{PH}{QK} = \frac{HQ}{KR} \therefore \frac{y-y_1}{y_1-y_2} = \frac{x_1-x}{x_2-x_1} \Rightarrow \frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \tag{3.9}$$

**Aliter:** Since points $P(x, y), Q(x_1, y_1)$ anad $R(x_2, y_2)$ are collinear, therefore

$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \Rightarrow y - y_1 = \frac{y_1-y_2}{x_1-x_2}(x - x_1).$

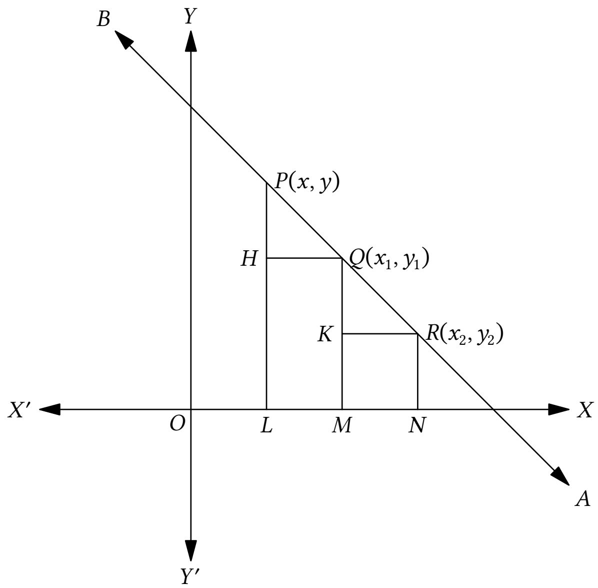



Figure 3.9: Two point form of a straight line

**Aliter:** Let the equation of straight line be $y = mx + c$. Since it passes through $(x_1, y_1)$ and $(x_2, y_2)$, therefore $y_1 = mx_1 + c, y_2 = mx_2 + c \Rightarrow m = \frac{y_1-y_2}{x_1-x_2}$. Now $c = y_1 - mx_1 = y_1 - (y_1 - y_2)(x_1 - x_2)x_1$

Substituting we get $y - y_1 = \frac{y_1-y_2}{x_1-x_2}(x - x_1)$.

### 3.5.4 Intercept Form

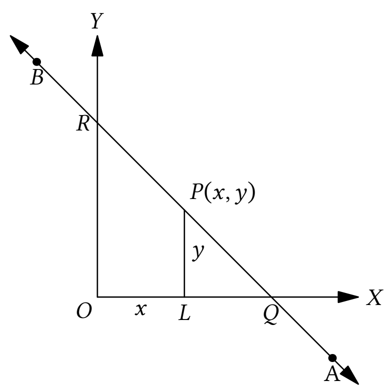



Figure 3.10: Intercept form of a straight line

We will find the equation of the straight line which cuts the intercept $a$ and $b$ at $x$-axis and $y$-axis respectively. See Figure 3.10.

Let $AB$ be a line which cuts intercept $a$ and $b$ on $x$-axis and $y$-axis respectively. Let line $AB$ cut $x$-axis at $Q$ and $y$-axis at $R$, then $OQ = a$ and $OR = b$.

Let $P(x, y)$ be any point on line $AB$, then $OL = x$ and $PL = y$.

Now in similar $\triangle\, QLP$ and $\triangle\, OQR$, we have

$\frac{QL}{QO} = \frac{PL}{OR}$ or $\frac{a-x}{a} = \frac{y}{b}$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1. \tag{3.10}$$

**Aliter:** Since $OQ = a$ and $OR = b$, therefore $Q \equiv (a, 0)$ and $R \equiv= (0, b)$. Since points $P(x, y), Q(a, 0)$ and $R(0, b)$ are collinear, therefore,

$\begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{vmatrix} = 0 \Rightarrow bx + ay = ab$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1. \tag{3.11}$$

**Aliter:** $\Delta QOR = \Delta POQ + \Delta PRO \Rightarrow \frac{1}{2}ab = \frac{1}{2}ay + \frac{1}{2}bx \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$.

### 3.5.5 Normal Form

We will find the equation of the straight line upon which the length of the perpendicular from the origin is $p$ and this normal makes an angle $\alpha$ with the positive direction of $x$-axis.

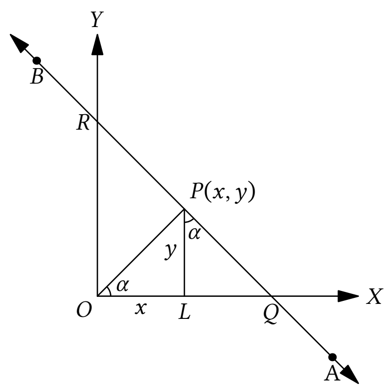



Figure 3.11: Normal form of a straight line

Let $AB$ be a line and $OP$ be perpendicular from the origin to $AB$. Let $OP = p$ and $\angle POX = \alpha$(the point can be different than $P$ but in this diagram $P$ is such that $OP$ is perpendicular to $AB$).

Let $P(x, y)$ be any point on the line $AB$. From $P$ we draw $PL$ perpendicular to $OX$. Let line $AB$ cut $x$-axis and $y$-axis at $Q$ and $R$ respectively.

Since $P \equiv (x, y) \therefore OL = x, PL = y$. Since $\angle POQ = \alpha \therefore \angle PQO = 90^{\circ} - \alpha, \angle LPQ = \alpha$

From $\triangle PLQ, LQ = PL \tan\alpha = y \tan\alpha$. Now $OP = OQ \cos\alpha = (OL + LQ)\cos\alpha$

$= (x + y\tan\alpha)\cos\alpha$

$$x\cos\alpha + y\sin\alpha = p. \tag{3.12}$$

**Aliter:** $OQ = p\sec\alpha$ and since $\angle ORP = \alpha, OR = p\csc\alpha$

Hence, $Q \equiv (p\sec\alpha, 0), R \equiv (0, p\csc\alpha)$

Now the points $P(x, y), Q(p\sec\alpha, 0)$ and $R(0, p\csc\alpha)$ are collinear, therefore,

$$\begin{vmatrix} x & y & 1 \\ p\sec\alpha & 0 & 1 \\ 0 & p\csc\alpha & 1 \end{vmatrix} = 0 \Rightarrow x(-p\csc\alpha) - yp\sec\alpha + 1.p^2\sec\alpha\csc\alpha = 0$$

$$\Rightarrow x\cos\alpha + y\sin\alpha = p. \tag{3.13}$$

### 3.5.6 Distance Form or Parametric Form

We will find the equation of the straight line passing through the point $(x_1, y_1)$ and making an angle $\theta$ with the positive direction of $x$-axis.

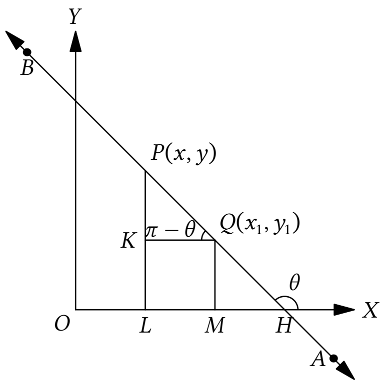



Figure 3.12: Distance form or parametric form of a straight line

Let $AB$ be a line which passes through the point $Q(x_1, y_1)$ and makes an angle $\theta$ with the positive direction of $x$-axis.

Let $P(x, y)$ be a point on the line $AB$.

From $P$ and $Q$ we draw $PL$ and $QM$ perpendicular on $x$-axis. From $Q$, we draw $QK$ perpendicular to $PL$.

$\because P \equiv (x, y) \therefore OL = x, PL = y$. $\because Q \equiv (x_1, y_1) \therefore OM = x_1, QM = y_1$.

$\because \angle BHX = \theta \therefore \angle BHO = \pi - \theta \therefore PQK = \pi - \theta$

Let $PQ = r$. From $\triangle PKQ, \cos(\pi - \theta) = \frac{KQ}{PQ} \Rightarrow -\cos\theta = \frac{x - x_1}{r} \Rightarrow \cos\theta = \frac{x - x_1}{r} \Rightarrow$ $\frac{x - x_1}{\cos\theta} = r$

Also, $\sin(\pi - \theta) = \frac{PK}{PQ} \Rightarrow \sin\theta = \frac{y-y_1}{r} \Rightarrow \frac{y-y_1}{\sin\theta} = r$

Thus,

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r. \tag{3.14}$$

**Corollary:** From Equation (3.14) we can say that

$$x = x_1 + r\cos\theta, y = y_1 + r\cos\theta \tag{3.15}$$

which is parametric form of a straight line.

If $Q(x_1, y_1)$ be a point on a line $AB$, which makes an angle $\theta$ with the positive direction of $x$-axis, then there will be two points on line $AB$ and their coordinates will be $(x_1 + r\cos\theta, y_1 + r\sin\theta)$ and $(x_1 - r\cos\theta, y_1 - r\sin\theta)$. These two points will be on two opposite sides of $Q$ on the line $AB$.

### 3.5.7 General Euqation of a Straight Line

We will show that the general equation of first degree in $x$ and $y$ always represent a straight line.

Let the general equation of first degree in $x$ and $y$ be $ax + by + c = 0$.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on this line. Then

$ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$.

Multiplying the two equations by $m$ and $n$ respectively and adding gives us

$a(mx_1 + nx_2) + b(my_1 + by_2) + c(m + n) = 0$

$\Rightarrow \frac{a(mx_1+nx_2)}{m+n} + \frac{b(my_1+ny_2)}{m+n} + c = 0$

Thus, point $\left(\frac{mx_1+nx_2}{m+n}, \frac{my_1+ny_2}{m+n}\right)$ lie on the line we have chosen.

Since $m$ and $n$ are arbitrary numbers, therefore, each point on the line $AB$ will lie on this locus. Hence, the general equation of a straight line is

$$ax + by + c = 0. \tag{3.16}$$

**Aliter:** Let the general equation of first degree in $x$ and $y$ be $ax + by + c = 0$.

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be any three points on the above line. Then

$ax_1 + by_1 + c = 0, ax_2 + by_2 + c = 0, ax_3 + by_3 + c = 0$

$\Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Thus, $A, B, C$ are collinear. Thus, the equation $ax + by + c = 0$ is equation of a straight line.

**Converse:** Every straight line may be represented by a first degree equation in $x$ and $y$.

We know that through two points one and only one straight line can be drawn.

Let $A(x_1, y_2)$ and $B(x_2, y_2)$ be two points on the straight line and let $P(x, y)$ be any point on it. Now since $P, A, B$ are collinear

$$\Rightarrow \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \Rightarrow (y_1 - y_2)x - (x_1 - x_2)y + x_1y_2 - x_2y_1 = 0$$

which is of the form $ax + by + c = 0$, where $a = y_1 - y_2, b = x_2 - x_1, c = x_1y_2 - x_2y_1$.

Since the equation $ax + by + c = 0$ is satisfied by all points on the line $AB$ and it will not be satisfied by the coordinates of any point which does not lie on the line $AB$, hence, it represents the equation of line $AB$.

## 3.6 Representing General Equation in Standard Forms

### 3.6.1 Slope Intercept Form

We will reduce the equattion $Ax + By + C = 0$ to $y = mx + c$

Given equation is $Ax + By + C = 0 \Rightarrow y = -\frac{A}{B}x - \frac{C}{B}$, which is of the form $y = mx + c$

Comparing we have $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$.

### 3.6.2 Intercept Form

We will reduce the equation $Ax + By + C = 0$ to $\frac{x}{a} + \frac{y}{b} = 1$.

This reduction is possible only if $C \neq 0$.

Given equation is $Ax + By + C = 0 \Rightarrow -\frac{A}{C}x - \frac{B}{C}y = 1$, where $C \neq 0$

$\Rightarrow \frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1$, which is of the form $\frac{x}{a} + \frac{y}{b} = 1$, where $a = -\frac{C}{A}, b = -\frac{C}{B}$.

### 3.6.3 Normal Form

We will reduce the equation $Ax + By + C = 0$ to the form $x\cos\alpha + y\sin\alpha = p$ where $\alpha$ is the angle made by the perpendicular on the line from origin and $p$ is the length of the perpendicular.

Given equation is $Ax + By = -C$

**Case I:** When $C < 0$ i.e. $-C > 0$.

Dividing both sides by $\sqrt{A^2 + B^2}$ gives us

$\frac{A}{\sqrt{A^2+B^2}}x + \frac{B}{\sqrt{A^2+B^2}}y = -\frac{C}{\sqrt{A^2+B^2}}$, which is of the form $x\cos\alpha + y\sin\alpha = p$.

**Case II:** When $C > 0$ i.e. $-C < 0$.

Given equation can be written as $-Ax - By = C$

$\Rightarrow -\frac{A}{\sqrt{A^2+B^2}}x - \frac{B}{\sqrt{A^2+B^2}}y = \frac{C}{\sqrt{A^2+B^2}}$, which is of the form $x\cos\alpha + y\sin\alpha = p$.

## 3.7 Angle between Two Straight Lines

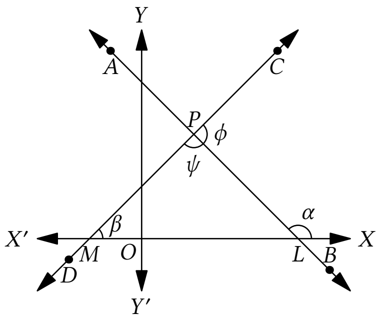



Figure 3.13: Angle between two straight lines

We will find the angle between two straight lines given by the equations $y = m_1x + c_1$ and $y = mx_2 + c_2$.

Let $AB$ and $CD$ be two given lines whose equations are $y = m_1x + c_1$ and $y = mx_2 + c_2$.

Let $AB$ and $CD$ cut each other at $P$ and they cut the $x$-axis at points $L$ and $M$ respectively.

Let $\angle MPL = \psi$ and $\angle ALX = \alpha, \angle DMX = \beta$. Now since line $AB$ makes an angle $\alpha$ with $x$-axis and its slope is $m_1$. $\therefore m_1 = \tan\alpha$.

Again since $CD$ makes an angle $\beta$ with $x$-axis and its slope be $m_2$. $\therefore m_2 = \tan\beta$.

From the figure $\alpha = \psi + \beta \Rightarrow \psi = \alpha - \beta$

$\tan\psi = \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{m_1 - m_2}{1 + m_1m_2}$

If $\angle PLD = \varphi$ then $\psi + \varphi = \pi \Rightarrow \varphi = \pi - \psi$

$\therefore \tan\varphi = -\tan\psi = -\frac{m_1m_2}{1 + m_1m_2}$

Since $\psi$ and $\varphi$ are the two angles between the lines $AB$ and $CD$, it follows that the angle $\theta$ between these two lines is given by

$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2} \Rightarrow \theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m - 1m_2}\right|, \pi - \tan^{-1}\left|\frac{m_1 - m_2}{1 + m - 1m_2}\right|$

If $\theta$ is the acute angle between the lines then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|. \tag{3.17}$$

**Notes:**

- There are two angles between two non-perpendicular lines. One of them is acute and the other one is obtuse and their sum is $180^\circ$. Thus, if acute angle $\theta$ between the two lines is known then the other angle will be $180^\circ - \theta$.
- The formula $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2}$ is valid only when $m_1$ and $m_2$ are defined.
- If both the lines are perpendicular to $x$-axis then the angle between them is $0^\circ$.

- If one of the lines is perpendicular to $x$-axis and the other line makes an angle of $\theta$ with positive direction of $x$-axis then the angle between them is $|90^{\circ} - \theta|$.

### 3.7.1 Parallelism and Perpendicularity

Our equation for angle between two straight lines is $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2}$. If the lines are parallel then the value of $\theta$ would be 0. Thus, $m_1 = m_2$. For special case when two lines are parallel to $y$-axis i.e. $m_1 = m_2 = \infty$ then both the lines are taken as parallel.

For lines to be perpendicular then $\tan\theta = \infty \Rightarrow 1 + m_1 m_2 = 0 \Rightarrow m_1 m_2 = -1$. When either of $m_1$ or $m_2$ is not defined then one of them has to be parallel to $x$-axis while the other has to be parallel to $y$-axis.

#### 3.7.1.1 To find the angle between straight lines $a_1x + b_1y + c = 0$ and $a_2x + b_2y + c = 0$

Writing the equation in slope intercept form we find that the slopes are given by $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$

If $\theta$ be the angle between the two lines then $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \frac{a_1}{b_1}\cdot\frac{a_2}{b_2}} = \pm\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$

$\theta = \tan^{-1}\left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$ or $\theta = \pi - \tan^{-1}\left|\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right|$

Clearly, the lines will be parallel if $\theta = 0$ i.e.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}. \tag{3.18}$$

and the lines will be perpendicular if

$$a_1a_2 + b_1b_2 = 0. \tag{3.19}$$

## 3.8 Lines Parallel to Another Line

We will find equation of a line parallel to another line in standard form i.e. $ax + by + c = 0$.

Let the line parallel to give line is $lx + my + n = 0$

From Equation (3.18) these two lines will be parallel if $\frac{l}{a} = \frac{m}{b} = p$ (say) then $l = ap, m = bp$

which makes the equation of the line as $ax + by + \frac{n}{p} = 0$ or $ax + by + k = 0$, where $k = \frac{n}{p}$.

Thus, equation of any line parallel to $ax + by + c = 0$ is given by

$$ax + by + k = 0. \tag{3.20}$$

## 3.9 Lines Perpendicular to Another Line

We will find equation of a line perpendicular to another line in standard form i.e. $ax + by + c = 0$.

Let the line perpendicular to give line is $lx + my + n = 0$

From Equation (3.19) these two lines will be parallel if $al + bm = 0$ or $\frac{l}{b} = -\frac{m}{a} = p$ (say)

Thus, $bpx - apy + n = 0 \Rightarrow bx - ay + \frac{n}{p} = 0 \Rightarrow bx - ay + k = 0$, where $k = \frac{n}{p}$

Thus, equation of any line perpendicular to $ax + by + c = 0$ is given by

$$bx - ay + k = 0. \tag{3.21}$$

## 3.10 Point of Intersection

We will find the point of intersection of two lines in standard form i.e. between the lines given by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Let $P(\alpha, \beta)$ is the point of intersection of these two lines then $a_1\alpha + b_1\beta + c_1 = 0$ and $a_2\alpha + b_2\beta + c_2 = 0$

By cross-multiplication we have $\frac{\alpha}{b_1c_2 - b_2c_1} = \frac{\beta}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$

Thus, the point of intersection is given by $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$.

As you can see in case of parallel lines $a_1b_2 - a_2b_1 = 0$ and then $\alpha$ and $\beta$ are not defined.

In this case neither $c_1a_2 - c_2a_1$ nor $b_1c_2 - b_2c_1$ will be zero, otherwise $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ i.e. the two lines will be coincident or same.

### 3.10.1 Line Passing through Point of Intersection of Two Lines

We will find the general equation of a straight line passing through point of intersection of two straight lines in standard form i.e. lines given by $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

We consider the straight line in first degree equation $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$.

Let $P(\alpha, \beta)$ be the point of intersection then $a_1\alpha + b_1\beta + c_1 = 0$ and $a_2\alpha + b_2\beta + c_2 = 0$ and $a_1\alpha + b_1\beta + c_1 + k(a_2\alpha + b_2\beta + c_2) = 0$

Clearly, the third equation is an equation of a line passing through $P$.

Thus, our desired equation is

$$a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0. \tag{3.22}$$

## 3.11 Concurrency of Three Straight Lines

We will find condition for concurrency of three lines in standard form i.e. the lines are given by $a_1x+b_1y+c_1=0, a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$.

Let $P(\alpha,\beta)$ be the point where these three lines meet. Then $(\alpha,\beta)$ will lie on all three lines. Thus, $a_1\alpha+b_1\beta+c_1=0$ and $a_2\alpha+b_2\beta+c_2=0$

Using cross-multiplication gives us $\frac{\alpha}{b_1c_2-b_2c_1}=\frac{\beta}{c_1a_2-c_2a_1}=\frac{1}{a_1b_2-a_2b_1}$

Thus, the point of intersection is given by $\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1},\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$.

Now this point will lie on the third line and therefore will satisfy the equation for third line. Hence,

$a_3\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}+b_3\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}+c_3=0.$

Thus, the required condition for concurrency of three lines is give by

$$a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0. \tag{3.23}$$

**Aliter:** Equations of given lines are $a_1x+b_1y+c_1=0, a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$.

Let $P(\alpha,\beta)$ be the point where these three lines meet. Then $(\alpha,\beta)$ will lie on all three lines. Thus, $a_1\alpha+b_1\beta+c_1=0, a_2\alpha+b_2\beta+c_2=0$ and $a_3\alpha+b_3\beta+c_3=0$

Eliminating $\alpha$ and $\beta$ we can write that

$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=0$, which is the required condition. Upon expansion the determinant becomes $a_3(b_1c_2-b_2c_1)+b_3(c_1a_2-c_2a_1)+c_3(a_1b_2-a_2b_1)=0$.

**Corollary:** The straight lines $a_1x+b_1y+c_1=0, a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$ will be concurrent if and only if there exists three constants $l,m,n$ (not all zero at the same time) such that $l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)=0$ identically i.e. for all values of $x$ and $y$.

Let there be three constants $l,m,n$ (not all zero at the same time) such that $l(a_1x+b_1y+c_1)+m(a_2x+b_2y+c_2)+n(a_3x+b_3y+c_3)=0$ for all values of $x$ and $y$.

Since $l,m,n$ are not zero at the same time let $n\neq 0$. Let $(\alpha,\beta)$ be the point of intersection of first two lines. Then

$a_1\alpha+b_1\beta+c_1=0$ and $a_2\alpha+b_2\beta+c_2=0$

Since the given condition is true for all $x$ and $y$, therefore

$l(a_1\alpha+b_1\beta+c_1)+m(a_2\alpha+b_2\beta+c_2)+n(a_3\alpha+b_3\beta+c_3)=0$

$\Rightarrow l.0+m.0+n(a_3\alpha+b_3\beta+c_3)=0\Rightarrow a_3\alpha+b_3\beta+c_3=0$

Thus, the third line also passes through $(\alpha,\beta)$.

## 3.12 Two Sides of a Straight Line

Every line divides the plane in two regions. Any point which does not lie on the line can be only on one side of the straight line.

We will show that a point $(\alpha, \beta)$ will be on one or the other side of the line $ax + by + c = 0$ according as the expression $a\alpha + b\beta + c > 0$ or $< 0$.

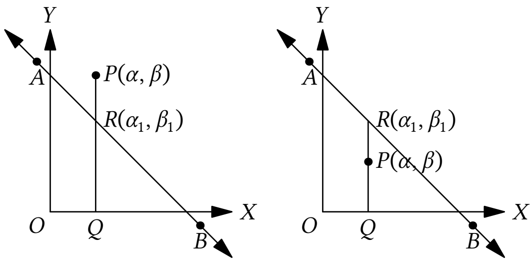



Figure 3.14:

Let $AB$ be the given line whose equation is $ax + by + c = 0$. Let $P(\alpha, \beta)$ be a point which does not lie on this line.

From $P$ we draw $PQ$ perpendicular on $x$-axis. Let $PQ$ cut line $AB$ at $R$. Clearly, $x$ coordinate of $P$ and $R$ are same.

Let $R \equiv (\alpha_1, \beta_1)$. Since it lies on the line, therefore, $a\alpha_1 + b\beta_1 + c_1 = 0$.

When $b > 0$, on left side of the Figure 3.14 $PQ > RQ \therefore \beta > \beta_1$ or $b\beta > b\beta_1$ or $a\alpha + b\beta + c > a\alpha + b\beta_1 + c$ or $a\alpha + b\beta + c > 0$

Similarly, for right side of the figure we can estanlish $a\alpha + b\beta + c < 0$

When $b < 0$, on left side of the Figure 3.14 $PQ > RQ \therefore \beta > \beta_1$ or $b\beta < b\beta_1$ or $a\alpha + b\beta + c < a\alpha + b\beta_1 + c$ or $a\alpha + b\beta + c < 0$

Similarly, for right side of the figure we can estanlish $a\alpha + b\beta + c > 0$

Thus, we see that $a\alpha + b\beta + c > 0$ or $< 0$ according as the point $P(\alpha, \beta)$ lies on one or the other side of the line $ax + by + c = 0$.

**Corollary:** It follows from previous article that two points $(x_1, y_1)$ and $(x_2, y_2)$ will lie on the same side or opposite side of the line $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign or opposite sign.

## 3.13 Length of a Perpendicular

We will find the length of the perpendicular from the point $(\alpha, \beta)$ to the line $ax + by + c = 0$.

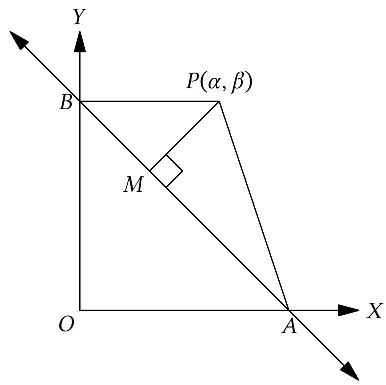



Figure 3.15: Length of a perpendicular

Let the given line be $ax + by + c = 0$ and given point be $P(\alpha, \beta)$. We have to find the length of the perpendicular from the point $P(\alpha, \beta)$ on line $AB$.

We draw $PM$ perpendicular to $AB$ and join $PA$ and $PB$. Let $PM = p$.

Putting $y = 0$ in the equation for the given line gives us $ax + c = 0 \Rightarrow x = -\frac{c}{a}$ $\therefore$ $A \equiv \left(-\frac{c}{a}, 0\right)$

Putting $x = 0$ in the equation for the given line gives us $by + c = 0 \Rightarrow y = -\frac{c}{b}$ $\therefore$ $B \equiv \left(0, -\frac{c}{b}\right)$

Now $\Delta PAB = \left|\frac{1}{2}\left[\alpha\left(0 + \frac{c}{b}\right) - \frac{c}{a}.\left(-\frac{c}{b} - \beta\right) + 0(\beta - 0)\right]\right| = \frac{1}{2}\left|\frac{c}{ab}\right||a\alpha + b\beta + c|$

Again $\Delta PAB = \frac{1}{2}.AB.PM = \frac{1}{2}\sqrt{\left(-\frac{c}{a} - 0\right)^2 + \left(0 + \frac{c}{b}\right)^2}.p = \frac{1}{2}\left|\frac{c}{ab}\right|\sqrt{a^2 + b^2}.p$

Equaating two obtained equations for $\Delta PAB$ gives us the length of the perpendicular, which is

$$p = \frac{|a\alpha + b\beta + c|}{\sqrt{a^2 + b^2}}. \tag{3.24}$$

**Aliter:** Let $PM$ make an angle $\theta$ with the positive direction of $x$-axis, then the equation of $PM$ in distance form is $\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta} = r$

Coordinates of any point on $PM$ at a distance $r$ from $P(\alpha, \beta)$ will be $(\alpha + r\cos\theta, \beta + r\sin\theta)$.

Since $PM = p$, therefore coordinates of $M$ is $(\alpha + p\cos\theta, \beta + p\sin\theta)$, which will lie on the line $ax + by + c = 0$, thus,

$a(\alpha + p\cos\theta) + b(\beta + p\sin\theta) + c = 0 \Rightarrow p(a\cos\theta + b\sin\theta) = -(a\alpha + b\beta + c)$

Now slope of $AB = -\frac{a}{b}$ $\therefore$ slope of $PM = \frac{b}{a}$ $\therefore \tan\theta = \frac{b}{a}$ or $\frac{a}{\cos\theta} = \frac{b}{\sin\theta} = k$ (say) $\therefore a = k\cos\theta, b = k\sin\theta$

$\Rightarrow a^2 + b^2 = k^2 \therefore k = \pm\sqrt{a^2 + b^2} \therefore a = \pm\sqrt{a^2 + b^2}\cos\theta \Rightarrow \cos\theta = \pm\frac{a}{\sqrt{a^2+b^2}}$.

Similarly, $\sin\theta = \pm\frac{b}{\sqrt{a^2+b^2}}$

Putting the values of $\cos\theta$ and $\sin\theta$ in the equation $p(a\cos\theta + b\sin\theta) = -(a\alpha + b\beta + c)$ gives us

$p = -\mp\frac{a\alpha+b\beta+c}{\sqrt{a^2+b^2}}$

Since $p$ is positive, therefore,

$$p = \frac{|a\alpha + b\beta + c|}{\sqrt{a^2+b^2}}. \tag{3.25}$$

**Aliter(Calculus Method):**

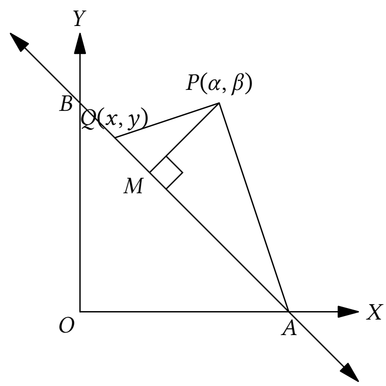



Figure 3.16: Length of a perpendicular

Let $Q(x, y)$ be any point on the line $AB$. Now $PQ$ will be the length of the perpendicular if $PQ$ is minimum. Hence, length of the perpendicular from $P$ on $AB$ will be the least value of $PQ$ when the point $Q(x, y)$ varies.

Let $PQ^2 = z$. Also, $PQ$ will be least if and only if $PQ^2$ i.e. $z$ is least.

Now $z = (x-\alpha)^2 + (y-\beta)^2$

Since $Q(x, y)$ lies on the line $ax + by + c = 0$, therefore, $y = -\frac{ax+c}{b}$

$\Rightarrow z = (x-\alpha)^2 + \left(-\frac{ax+c}{b} - \beta\right)^2 = (x-\alpha)^2 + \left(-\frac{ax+c}{b} - \beta\right)^2$

$\frac{dz}{dx} = 2(x-\alpha) + \frac{1}{b^2}.2(ax + b\beta + c)$

For maxima and minima of $z$, $\frac{dz}{dx} = 0$

$\Rightarrow x - \alpha + \frac{a}{b^2}(ax + b\beta + c) = 0 \Rightarrow x - \alpha + \frac{a}{b^2}(-by + b\beta) = 0$

$\Rightarrow \left(\frac{y-\beta}{x-\alpha}\right) = \frac{b}{a} \Rightarrow \frac{y-\beta}{x-\alpha}.\left(-\frac{a}{b}\right) = -1$

$\therefore$ Slope of line $PQ$.Slope of line $AB = -1$

Thus, when $\frac{dz}{dx} = 0, PQ \perp AB$

Since maximum length of $PQ$ is not possible as it will be $\infty$ (tend to $\infty$) and hence $\frac{dz}{dx} = 0$ gives the minimum length of $PQ$.

Length of perpendicular $z = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$

But if $\frac{dz}{dx} = 0$ then $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \left(a(x-\alpha) + \frac{b(y-\beta)}{a^2+b^2}\right)$

$= \frac{ax+by+c-(a\alpha+b\beta+c)}{a^2+b^2} = -\frac{a\alpha+b\beta+c}{a^2+b^2}$

$x - \alpha = -\frac{a(a\alpha+b\beta+c)}{a^2+b^2}$ and $y - \beta = -\frac{b(a\alpha+b\beta+c)}{a^2+b^2}$

$z = \frac{|a\alpha+b\beta+c|}{\sqrt{a^2+b^2}} = p$

**Note:** Length of perpendicular from origin on line $ax + by + c = 0$ is $\frac{|c|}{\sqrt{a^2+b^2}}$

## 3.14 Bisectors of Angles between Straight Lines

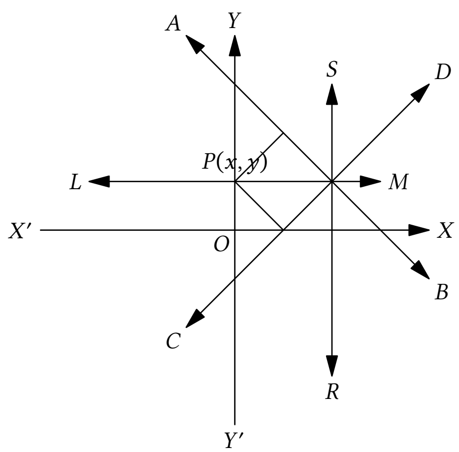



Figure 3.17: Bisectors of angles between straight lines

We will find the equation of the bisectors of the angles between the straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Let the given lines be $AB$ and $CD$ whose equations are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Let $LM$ and $RS$ be the two bisectors of the angles between $AB$ and $CD$. Let $P(x, y)$ be the point on any bisector. Since $P$ lies on a bisector, therefore, the lengths of perpendiculars on two lines will be equal.

Thus, the length of perpendicular from $P$ to $AB$ will be equal to the length of the perpendicular from $P$ to $CD$.

$\Rightarrow \frac{|a_1x+b_1y+c_1|}{\sqrt{a_1^2+b_1^2}} = \frac{|a_2x+b_2y+c_2|}{\sqrt{a_2^2+b_2^2}}$

Thus,

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \tag{3.26}$$

are the equations of two bisectors.

**Note:** If $P(x, y)$ is taken on the bisector of the angle which contains the origin then either $O(0, 0)$ and $P(x, y)$ will lie on the same sides of two lines. Thus,

$a_1x + b_1y + c_1 > 0$ and $a_2x + b_2y + c_2 > 0$

or $O(0, 0)$ and $P(x, y)$ will lie on the oppposite side of the two lines i.e.

$a_1x + b_1y + c_1 < 0$ and $a_2x + b_2y + c_2 < 0$

Then equation of bisectors will be

$\frac{|a_1x+b_1y+c_1|}{\sqrt{a_1^2+b_1^2}} = \frac{|a_2x+b_2y+c_2|}{\sqrt{a_2^2+b_2^2}}$

i.e. $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ when both $a_1x + b_1y + c_1$ and $a_2x + b_2y + c_2$ are positive

or $-\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ when both $a_1x + b_1y + c_1$ and $a_2x + b_2y + c_2$ are negative.

Thus, in both the cases equation of the bisector containing the origin when $c_1$ and $c_2$ are both positive is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \tag{3.27}$$

When both $c_1$ and $c_2$ are positive, then the equation of the bisector of the angle between the lines which does not contain the origin is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \tag{3.28}$$

The two bisectors are perpendicular to each other.

### 3.14.1 Finding Bisector of the Acute and Obtuse Angles

To find the bisector of the acute and obtuse angles take any line out of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ and any of the bisectors obtained. Let $\theta$ be the angle between them. Find $|\tan\theta|$. The bisector considered will be the bisector of the acute angle or obtuse angle between the lines according as $\theta < 45^\circ$ or $\theta > 45^\circ$ i.e. according as $|\tan\theta| < 1$ or $> 1$.

**Aliter:** Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $c_1 > 0$ and $c_2 > 0$

Then the equation $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ is the equation of the bisector of the acute or obtuse angle between the lines according as $a_1a_2 + b_1b_2 < 0$ or $> 0$.

Slope of bisector $= m_1 = -\frac{a_1\sqrt{a_2^2+b_2^2} - a_2\sqrt{a_1^2+b_1^2}}{b_1\sqrt{a_2^2+b_2^2} - b_2\sqrt{a_1^2+b_1^2}}$ and slope of first line is $m_2 = -\frac{a_1}{b_1}$.

Angle between first line and bisector is $\tan\theta = \frac{m_1-m_2}{1+m_1m_2} = \frac{a_1b_2-a_2b_1}{\sqrt{(a_1b_2-a_2b_1)^2+(a_1a_2+b_1b_2)^2}-(a_1a_2+b_1b_2)}$

Let $\alpha = a_1b_2 - a_2b_1$ and $\beta = a_1a_2 + b_1b_2$ then $|\tan\theta| = \frac{|\alpha|}{\sqrt{\alpha^2+\beta^2-\beta}}$

If $\beta < 0$ then $\sqrt{\alpha^2+\beta^2-\beta} > |\alpha| - \beta > |\alpha|$, therefore, $|\tan\theta| < 1$ and hence, the bisector is the bisector of acute angle between the lines.

If $\beta > -0$ then $\sqrt{\alpha^2+\beta^2} \leq |\alpha| + |\beta| \Rightarrow \sqrt{\alpha^2+\beta^2-\beta} \leq |\alpha|$, therefore, $|\tan\theta| > 1$, and hence, the bisector is the bisector of obtuse angle between the lines.

Similarly, we can show that the equation $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = -\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$ is the equation of the bisector of the acute or obtuse angles according as $a_1a_2 + b_1b_2 > 0$ or $< 0$.

## 3.14.2 Bisectors Between Lines Containing a Given Point

Let the equations be of the lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

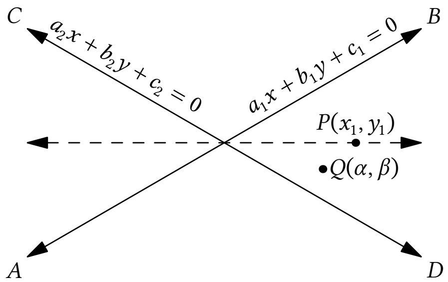
Let $(\alpha, \beta)$ be a given point. If $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of the same sign then equation of the bisector of the angle containing the point $(\alpha, \beta)$ is

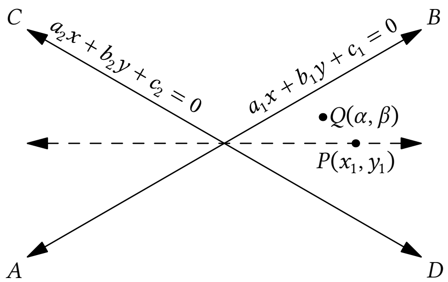
$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \tag{3.29}$$

Let $(x_1, y_1)$ be a point on the bisector then $\frac{a_1x_1+b_1y_1+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x_1+b_2y_1+c_2}{\sqrt{a_2^2+b_2^2}}$

Since $\sqrt{a_1^2 + b_1^2}$ and $\sqrt{a_2^2 + b_2^2}$ are both posiitive, therefore, $a_1x_1 + b_1y_1 + c_1$ and $a_2x_1 + b_2y_1 + c_1$ are of the same sign.

**Case I:** When both $a_1x_1 + b_1y_1 + c_1$ and $a_2x_1 + b_2y_1 + c_1$ are positive.




Since $a_1\alpha+b_1\beta+c_1$ and $a_1x_1+b_1y_1+c_1$ are both posiitive, therefore, points $P(x_1, y_1)$ and $Q(\alpha, \beta)$ will lie on the same side of the line $a_1x+b_1y+c_1=0$. Agains since $a_2\alpha+b_2\beta+c_2$ and $a_2x_1+b_2y_1+c_2$ are both posiitive, therefore, points $P(x_1, y_1)$ and $Q(\alpha, \beta)$ will lie on the same side of the line $a_2x+b_2y+c_2=0$.

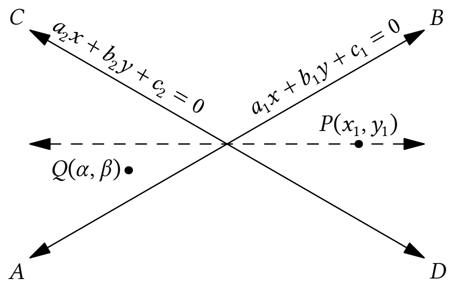
The figure will be like Figure 3.18.

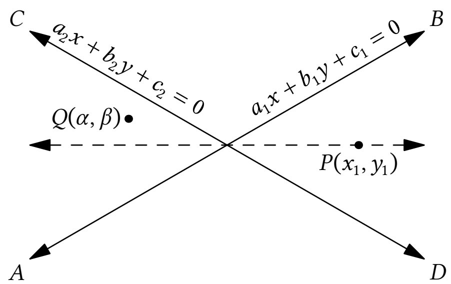
**Case II:** When both $a_1x_1+b_1y_1+c_1$ and $a_2x_1+b_2y_1+c_1$ are negative.

In this case $a_1\alpha+b_1\beta+c_1$ and $a_1x_1+b_1y_1+c_1$ are of opposite sign, therefore, points $P(x_1, y_1)$ and $Q(\alpha, \beta)$ will lie on opposite side of the line $a_1x+b_1y+c_1=0$.

Again since $a_2\alpha+b_2\beta+c_2$ and $a_2x_1+b_2y_1+c_2$ are of opposite sign, therefore, points $P(x_1, y_1)$ and $Q(\alpha, \beta)$ will lie on opposite side of the line $a_2x+b_2y+c_2=0$.

Thus, in this case figure will be as given below:




Similarly, if $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of opposite sign then the equation of the bisector of the angle containing the point $(\alpha, \beta)$ is

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}. \tag{3.30}$$

**Aliter:** Let the equations of lines $AB$ and $CD$ are $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, where $c_1$ and $c_2$ are positive.

These equations in normal form will be
$-\frac{a_1x}{\sqrt{a_1^2+b_1^2}} - \frac{b_1y}{\sqrt{a_1^2+b_1^2}} = \frac{c_1}{\sqrt{a_1^2+b_1^2}}$ and $-\frac{a_2x}{\sqrt{a_2^2+b_2^2}} - \frac{b_2y}{\sqrt{a_2^2+b_2^2}} = \frac{c_2}{\sqrt{a_2^2+b_2^2}}$
Let $\cos\alpha = -\frac{a_1}{\sqrt{a_1^2+b_1^2}}, \cos\beta = -\frac{a_2}{\sqrt{a_2^2+b^2}}, \sin\alpha = -\frac{b_1}{\sqrt{a_2^2+b_2^2}}, \sin\beta = -\frac{b_2}{\sqrt{a_2^2+b^2}}$
Now $\cos(\beta - \alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta = \frac{a_1a_2+b_1b_2}{\sqrt{a_1^2+b_1^2}\sqrt{a_2^2+b_2^2}}$

$\beta - \alpha$ will be acute or obtuse according as $a_1a_2 + b_1b_2 > 0$ or $< 0$

Now $\angle CAB$ will be acute or obtuse according as $\beta - \alpha$ is obtuse or acute i.e. according as $a_1a_2 + b_1b_2 < 0$ or $> 0$

But $\angle CAB$ contains the origin, therefore, origin will be contained in the acute or obtuse angle according as $a_1a_2 + b_1b_2 < 0$ or $> 0$

Hence, bisector of the angle between the lines will be bisector of the acute or obtuse angle according as origin lies in the acute or obtuse angle i.e. according as $a_1a_2 + b_1b_2 < 0$ or $> 0$.

## 3.15 Problems

1. Find the equation of the straight line cutting off an intercept 5 from the positive direction of $y$-axis, and inclined at angle $45^{\circ}$ to the $x$-axis.
2. Find the equation of the straight line passing through the point $(2, -3)$ and cutting off intercepts, equal but of opposite signs from the two axes.
3. Find the equation of the straight line which passes through the point $(-5, 4)$ and is such that the portion of it between the axes is doivided by the point in the ratio of $1 : 2$.
4. Find the normal form of the equation $x + y\sqrt{3} + 7 = 0$.
5. Find the equation of the straight line which passes through the points $(-1, 3)$ and $(4, -2)$.
6. Find the equation of the straight line cutting off intercept unity from the positive direction of the $y$-axis and inclines at $45^{\circ}$ to the $x$-axis.
7. Find the equation of the straight line cutting off intercept $-5$ from the $y$-axis and being equally inclined to the axes.
8. Find the equation of the straight line cutting off intercept 2 from the negative direction of $y$-axis and incllined at $30^{\circ}$ to $OX$.
9. Find the equation of the straight line cutting off intercept $-3$ from the $y$-axis and inclines at an angle $\tan^{-1}\frac{3}{5}$ to the $y$-axis.
10. Find the equation of the straight line cutting off intercepts 3 and 2 from the axes.
11. Find the equation of the straight line cutting off intercepts $-5$ and 6 from the axes.
12. Find the equation of the straight line which passes through the point $(5, 6)$ and has intercept on the axes equal in magnitude and both positive. Find the equation if intercepts are equal in magnitude but opposite in sign.
13. Find the equations of the straight lines which passes through the point $(1, -2)$ and cut off equal distances from the two axes.
14. Find the equation of the straight line which passes through the point $(x', y')$ and is such that the given point bisects the part intercepted between the axes.
15. Find the equation of the straight line which passes through the point $(-4, 3)$ and is such that the portion of it between the axes is divided by the point in the ratio $5 : 3$.
16. Find the equation of the straight line passing through the points $(0, 0)$ and $(2, -2)$.
17. Find the equation of the straight line passing through the points $(3, 4)$ and $(5, 6)$.
18. Find the equation of the straight line passing through the points $(-1, 3)$ and $(6, -7)$.
19. Find the equation of the straight line passing through the points $(0, -a)$ and $(b, 0)$.
20. Find the equation of the straight line passing through the points $(a, b)$ and $(a + b, a - b)$.

21. Find the equation of the straight line passing through the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

22. Find the equation of the straight line passing through the points $\left(at_1, \frac{a}{t_1}\right)$ and $\left(at_2, \frac{a}{t_2}\right)$.

23. Find the equation of the straight line passing through the points $(a\cos\varphi_1, a\sin\varphi_1)$ and $(a\cos\varphi_2, a\sin\varphi_2)$.

24. Find the equation of the straight line passing through the points $(a\cos\varphi_1, b\sin\varphi_1)$ and $(a\cos\varphi_2, b\sin\varphi_2)$.

25. Find the equation of the straight line passing through the points $(a\sec\varphi_1, b\tan\varphi_1)$ and $(a\sec\varphi_2, b\tan\varphi_2)$.

26. Find the equations og the sides of the triangle the coordinates of whose angular points are respectively $(1, 4), (2, -3)$ and $(-1, -2)$.

27. Find the equations of the sides of the triangle the coordinates of whose angular points are respectively $(0, 1), (2, 0)$ and $(-1, -2)$.

28. Find the equations of the diagonals of the rectangle the equations of whose sides are $x = a, x = a', y = b$, and $y = b'$.

29. Find the equation of the straight line which bisects the distance between the points $(a, b)$ and $(a', b')$ and also bisects the distance between the points $(-a, b)$ and $(a', -b')$.

30. Find the equations of the straight lines which go through the origin and trisect the portion of the straight line $3x + y = 12$ which is intercepted between the axes of coordinates.

31. Find the equation of the straight line which makes an angle of $15^\circ$ with the positive direction of $x$-axis, and which cuts an intercept of length 4 on the negative direction of $y$-axis.

32. Find the equation of the straight line which cuts off an intercept 4 from the $x$-axis and makes an angle of $30^\circ$ with the $y$-axis.

33. Find the equation of the straight line which passes through the point $(1, 2)$ and makes an angle $\theta$ with the positive direction of $x$-axis where $\cos\theta = -\frac{1}{3}$.

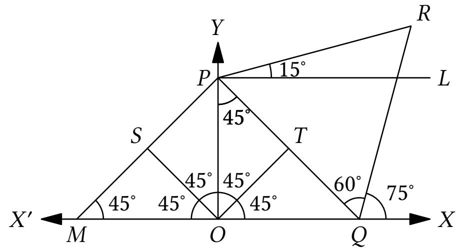
34. Find the equation of the line joining the points $(-1, 3)$ and $(4, -2)$.

35. A line through the point $A(2, 0)$ which makes an angle of $30^\circ$ with the positive direction of $x$-axis is rotated about $A$ in clockwise direction through an angle of $15^\circ$. Find the equation of the straight line in the new position.

36. Find the equation of the internal bisector of the $\angle BAC$ of the $\triangle ABC$ whose vertices $A, B, C$ are $(5, 2), (2, 3), (6, 5)$ respectively.

37. A rectangle has two opposite vertices at the point $(1, 2)$ and $(5, 5)$. If the other vertices lies on the line $x = 3$, find the equation of the sides of the triangle.

38. In the given figure $PQR$ is an equilateral triangle and $OSPT$ is a square. If $OT = 2\sqrt{2}$ units, find the equation of lines $OT, OS, SP, QR, PR$ and $PQ$.




39. If $D, E, F$ are three points on the sides $BC, AC$ and $AB$ of a $\triangle\, ABC$ such that $AD, BE$ and $CF$ are concurrent, then show thta $BD.CE.AF = DC.EA.FB$.

40. Find the coordinates of the vertices of a square inscribed in a triangle with vertices $A(0,0), B(2,1)$ and $C(3,0)$; given that two of its vertices are on the side $AC$.

41. Transform equation $\sqrt{3}y - 3x = 3$ to the slope intercept form and also find the angle, which this straight line makes with the $x$-axis.

42. Find the equation of the straight line which cuts of an intercept of 7 on $y$-axis and has the slope 3.

43. Find the equation of the line, which makes an angle of $75^{\circ}$ with $x$-axis and cuts an intercept of length 3 on the positive direction of $y$-axis.

44. Find the equation of the straight lines which cuts off an intercept $-5$ from the $y$ -axis and makes an angle of $\sin^{-1}\frac{12}{13}$ with the $x$-axis.

45. Find the equation of the straight line which is parallel to $x$-axis at a distance of 5 units from it.

46. Find the equation of the straight line which is parallel to $y$-axis at a distance of 4 units from it towards negative side of $x$-axis.

47. Find the equation of the straight lines which pass through $(5,3)$ and are respectively parallel and perpendicular to the $x$-axis.

48. Find the equation of the straight line which intercepts a length of 2 on the positive direction of $x$-axis and is inclined at $135^{\circ}$ with the positive direction of $y$-axis.

49. Find the equation of a straight line which cuts off an intercept 4 on the $x$-axis and has the slope 2.

50. Find the equation of the straight line passing through $(3,-2)$ and making an angle of $60^{\circ}$ with the positive direction of $y$-axis.

51. Find the slope of the line passing through the points $(3,4)$ and $(1,2)$. Also find its equation.

52. Find the equation of the straight line passing through the points $(a,b)$ and $(a + r\cos\theta, b + r\sin\theta)$.

53. If $(x,y)$ is a point on the straight line joining $(1,-3)$ and $(-4,2)$ show that $x + y + 2 = 0$.

54. Prove that the points $(1,4), (3,-2)$ and $(-3,16)$ are collinear. Also, find the the equation of this straight line.

55. If the points $(a, b), (a_1, b_1)$ and $(a - a_1, b - b_1)$ are collinear, show that the line joining them passes through the origin.
56. Find the value of $t$ for which $(1, 2), (-30)$ and $(t - 1, 3)$ will be collinear. Also find the equation of the straight line.
57. Show that the straight line passing through the points $(p, q + r)$ and $(q, r + p)$ also passes through the point $(r, p + q)$.
58. Find the equation of the straight line passes through the point which divides the line segment joining the points $(-1, 2)$ and $(4, -5)$ externally in the ratio $2 : 3$ and the point $(1, 2)$.
59. Find the equation of the side $BC$ of the $\triangle ABC$ whose vertices $A, B, C$ are $(-1, -2), (0, 1), (2, 0)$ respectively. Also find the equation of the median through $(-1, -2)$.
60. The vertices of a triangle are $(1, 2), (2, 3)$ and $(5, 4)$. Find the equation of its medians.
61. In what ratio does the line $x + y + 1 = 0$ divide the line segment joining the points $(2, 3)$ and $(-1, 4)$?
62. Find the ratio in which the line segment joining $(2, 3)$ and $(4, 1)$ is divided by the line joining $(1, 2)$ and $(4, 3)$.
63. The vertices of a $\triangle ABC$ are $(2, 2), (-1, -2)$ and $(1, -3)$. $D$ is a point on $BC$ such that $BD : DC = 2 : 1$. Find the ratio in which $AD$ divides the median through $B$.
64. For the straight line $y - \sqrt{3}x = 3$, find the intercept on $y$-axis and also the angle which the straight line makes with the $x$-axis.
65. Find the equation of the straight line which passes through the point $(3, 4)$ and whose intercept on $y$-axis is twice that on $x$-axis.
66. Find the equation of the straight line which passes through the point $(3, 4)$ and is such that the portion of it intercepted between the axes is divided by the point in the ratio $2 : 3$.
67. Find the equation of the straight line whose intercepts on $x$-axis and $y$-axis are respectively twice and thrice of those by the line $3x + 4y = 12$.
68. Find the equation of the straight line passing through the origin and the middle point of the intercept of the straight line $ax + by + c = 0$ between the axes.
69. Find the equation of the straight lines which pass through the origin and trisect the intercept of the line $3x + 4y = 12$ between the axes.
70. A straight line cuts intercepts from the axes of coordinates the sum of the reciprocal of which is a constant. Show that it always passes through a fixed point.
71. If the equal sides $AB$ and $AC$ of a right angled isosceles triqangle be produced to $P$ and $Q$ so that $BO.CQ = AB^2$, show that $PQ$ always passes through a fixed point.

72. Through the point $P(\alpha, \beta)$, where $\alpha\beta > 0$ the straight line $\frac{x}{a} + \frac{y}{b} = 1$ is drawn so as to form with coordinate axes a triangle of area $S$. If $ab > 0$, find the least value of $S$.

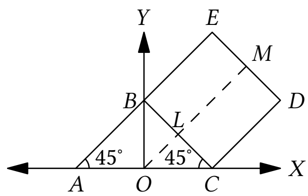
73. Find the equation of the straight line upon which the length of perpendicular from origin is $3\sqrt{2}$ units and this perpendicular makes an angle of $75^\circ$ with the positive direction of $x$-axis.

74. Find the equation of the straight line upon which the length of the perpendicular from the origin is 2 and the slope of the perpendicular is $\frac{5}{12}$.

75. A canal is $4\frac{1}{2}$ kms from a place and the shortest route from this place to canal is exactly north-east. A village is 3 kms north and 4 kms east from the place. Does it lie on the canal?

76. Find the equation of the straight lines which makes a triangle of area $96\sqrt{3}$ with the axes and perpendicular from the origin to it makes an angle of $30^\circ$ with $y$ -axis.

77.




In the given figure $ABC$ is a right-angled iscosceles triangle and $BCDE$ is a square. If $OC = 2$, find the equation of the sides $AB$ and $BC$ of $\triangle ABC$ and side $DE$ of the square.

78. Find the coordinates of the point where the line $\sqrt{3}x + y - 8 = 0$ meets the coordinate axes and also find the length of the perpendicular from the origin upon this line and the angle which this perpendicular makes with the $x$-axis.

79. Find the equations of the straight lines which pass through the point $(3, 2)$ and cut off intercepts $a$ and $b$ on the $x$ and $y$-axes such that $a - b = 2$.

80. Find the equation of the line which passes through $P(1, -7)$ and meets the axes at $A$ and $B$ respectively such that $4AP - 3BP = 0$, where $O$ is the origin.

81. Find the equation of the straight line which passes through the point $P(2, 6)$ and cuts the axes at the points $A$ and $B$ respectively such that $\frac{AP}{BP} = \frac{2}{3}$.

82. Find the equation of the straight line whose intercepts on the axes are twice the intercepts of the line $3x + 4y = 6$.

83. Find the equation of the straight line passing through $(2, 1)$ and bisecting the portion of the straight line $3x - 5y = 15$ lying between the axes.

84. Find the equations of the straight lines which pass through the origin and trisect the portion of the straight line $2x + 3y = 6$ which is intercepted between the axes.

85. Prove that the points $(5, 1), (11, 4)$ and $(1, -1)$ lie on a straight line and find its intercepts on the axes and between the axes.

86. Find the intercepts on the axes of the straight line passing through the points $(1,-3)$ and $(4,5)$.

87. The lnegth of the perpendicular from the origin to a line is 7 and the line makes an angle of $150^{\circ}$ with the positive direction of $y$-axis. Find the equation of the line.

88. Find the equation of the straight line upon which the length of the perpendicular from the origin is 2 and this perpendicular makes an angle of $30^{\circ}$ with the positive direction of $y$-axis.

89. Find the equation of the line which is at a distance 5 from the origin and the perpendicular from the origin to the line makes an angle of $60^{\circ}$ with the positive direction of $x$-axis.

90. Find the equation of the straight line upon which the length of the perpendicular from the origin is 6 and the gradient of the perpendicular is $\frac{3}{4}$.

91. Find the equation of the line joining the points $(1,2)$ and $(-3,1)$. Find its intercepts on the axes. If $p$ be the length of the perpendicular from the origin to the line, find the value of $p$.

92. Find the equation of the straight line which passes through the point $(3,2)$ and whose gradient is $\frac{3}{4}$. Find the coordinates of the points on the line that are 5 units away from the point $(3,2)$.

93. Find the direction in which a straight line must be drawn through the point $(1,2)$ so that its point of intersection wth the line $x+y=4$ is at a distance $\sqrt{\frac{2}{3}}$ from the point $(1,2)$.

94. If the straight line drawn through the point $P\left(\sqrt{3},2\right)$ makes an angle $\frac{\pi}{6}$ with the $x$-axis meets the line $\sqrt{3}x-4y+8=0$ at $Q$. Find the length of $PQ$.

95. Find the coordinates of the points at a distance $4\sqrt{2}$ units from the point $(02,3)$ in the direction making an angle of $45^{\circ}$ with the positive direction of $x$-axis.

96. A line joining two points $A(2,0)$ and $B(3,1)$ is rotated about $A$ in anticlockwise direction through an angle $15^{\circ}$. Find the equation of the line in new position. If $B$ goes to $C$ in the new position what will be the coordinates of $C$?

97. The extremeties of the diagonal of a sqaure are $(1,1),(-2,-1)$. Obtain the other two vertices and the equation of the other diagonal.

98. Show that if any line through the variable point $A(k+1,2k)$ meets the line $7x+y-16=0, 5x-y-8=0, x-5y+8=0$ at $B,C,D$ respectively $AC,AB$ and $AD$ are in H.P.

99. The center of a square is at the origin and one vertex is $A(2,1)$. Find the coordinates of other vertices of the square.

100. Show that if $A(x_1,y_1),B(x_2,y_2),C(x_3,y_3)$ are the vertices of a triangle, then the equation of the internal bisector of angle $A$ is given by $b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}+c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0$, where $b=AC$ and $c=AB$.

101. Find the coordinate of the point at a distance 6 units from the point $(1,1)$ in the direction making an angle of $60^\circ$ with the positive direction of $x$-axis.

102. Find the distance of the line $2x+y=3$ from the point $(-1,3)$ in the direction whose slope is one.

103. The straight line through $P(x_1,y_1)$ inclined at an angle $\theta$ with $x$-axis meets the line $ax+by+c=0$ in $Q$. Find the length of $PQ$.

104. A line through the point $A(2,0)$ which makes an angle of $30^\circ$ with the positive direction of $x$-axis is rotated about $A$ in anticlockwise direction through an angle $15^\circ$. Find the equation of the straight line in the new position.

105. The straight line $2x-y=5$ turns about the point on it where the ordinate is equal to the abcissa through an angle of $45^\circ$ in the anti-clockwise direction. Find the equation of the line in the new position.

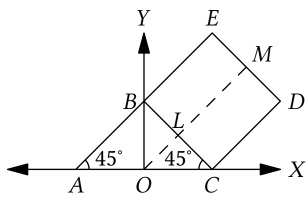
106. The straight line $x+2y=4$ is translated parallel to itself by 3 units in the sense of increasing $x$ and is then rotated by $30^\circ$ in the clockwise direction about the point where the shifted straight line cuts the $x$-axis. Find the equation of the new straight line in the new position.

107. $AB$ is a side of regular hexagon $ABCDEF$ and is of length a with $A$ as the origin and $AB$ and $AE$ as the $x$-axis and $y$-axis respectively. Find the equation of lines $AC, AF$ and $BE$.

108. A straight road is at a distance of $5\sqrt{2}$ miles froma place. The shortest distance of the road from the place is in $N-E$ direction. Do the following villages which (i) is 6 miles East and 4 miles and 4 miles North from the place, lie on the road or not. (ii) is 4 miles East and 3 miles north from the place, lie on the road or not?

109. A straight line $x-y+1=0$ cuts the $y$-axis at $A$. This line is rotated about $A$ in the clockwise direction by $75^\circ$. Fine the equation of the new straight line.

110. Find the equations of all sides of the isosceles $\triangle ABC$ and the side $BE$ and $CD$ of the square $BCDE$ in the figure where $OC=2$ units.




111. The mid-point of the line segment joining $(3,-1)$ and $(1,1)$ is shifted by two units(in the sense of increasing $y$) perpendicular to the line segment. Find the coordinates of the point in the new position.

112. The point $(1,1)$ is translated parallel to the line $2x=y$ in the first quadrant through a unit distance. Find the new position of the point.

113. The point $A(2,-1)$ is translated parallel to the line $x-y=3$ by a distance of 4 units. If the new position $A'$ of the point is in the (i) first quadrant (ii) third quadrant then find $A'$.

114. Two particles start from the same point $(2, -1)$, one moving 2 units along the line $x + y = 1$ and the other 5 units along the line $x - 2y = 4$. If the particles move towards increasing $y$, find their new positions and the distance between them.

115. One end of a thin elastic straight string is fixed at $A(4, -1)$ and the other end $B$ is at $(1, 2)$ in the unstretched condition. If the string is stretched to triple its length, find the coordinates of the other end in this stretched position.

116. The line $AB$ whose equation is $x - y = 2$ cuts the $x$-axis at $A$ and $B$ is $(4, 2)$. The line $AB$ is rotated about $A$ through $45^\circ$ in the anticlockwise direction. Find the new position of $B$ and the equation of the line in new position.

117. Let $xy$ plane be vertical. A particle dropped gently from $(-1, 1)$ in the plane rebounds on the floor returns to $\frac{2}{3}$rd of the height from which it has fallen. The equation of the line of intersection of $xy$ plane and the floor is $x + 2y = 3$. Find the highest position of the particle after one rebound.

118. A line is drawn through $A(4, -1)$ parallel to the line $3x - 4y + 1 = 0$. Find the coordinates of the two points on this line which are at a distance of 5 units from $A$.

119. Find the distance of the point $(3, 5)$ from the line $2x + 3y = 14$ measured parallel to the line $x - 2y = 1$.

120. Find the distance of the point $(2, 5)$ from the line $3x + y + 4 = 0$ measured parallel to the line $3x - 4y + 8 = 0$.

121. The point $(1, 3)$ and $(5, 1)$ are two opposite sides of a rectangle and the other two vertices lie on the line $y = 2x + c$. Find $c$ and other vertices.

122. A line is drawn from $(x', y')$ in the direction $\alpha$ with the $x$-axis, to meet $Ax + By + C = 0$. Prove that the lengeth is $\left|\frac{Ax'+By'+c}{A\cos\alpha+B\sin\alpha}\right|$.

123. Find the equation of the line passing through the point $P(1, 2)$ cutting the lines $x + y - 5 = 0$ and $2x - y = 7$ at A and $B$ respectively such that the H. M. of $PA$ and $PB$ is 10. Given that $A, B$ lie on the same side of $P$.

124. A straight line through the point $A(-2, 3)$ cuts the line $x + 3y = 9$ and $x + y + 1 = 0$ at $B$ and $C$ respectively. Find the equation of the line if $AB.AC = 20$.

125. A line which makes an acute angle $\theta$ with the positive direction of $x$-axis is drawn through the point $P(3, 4)$ to cut the curve $y^2 = 4x$ at $Q$ and $R$. Show that the lengths of the segments $PQ$ and $PR$ are numerical values of roots of the equation $r^2 \sin^2\theta + 4r(\sin\theta - \cos\theta) + 4 = 0$.

126. Show that if $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle then the equation of the median through $A$ is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$

127. Find the angle between the lines $x - 2y + 3 = 0$ and $3x + y - 1 = 0$.

128. Find the angle between the line $x + y = 3$ and the line joining the points $(1, 1)$ and $(-3, 4)$.

129. Find the value of $k$ so that the straight line $2x + 3y + 4 + k(6x - y + 12) = 0$ is perpendicular to the line $7x + 5y - 4 = 0$.
130. Prove that the line joining the middle points of the two sides of a triangle is parallel to the third side.
131. Find the values of $x$ and $y$ for which $A(2, 0), B(0, 2), C(0, 7)$ and $D(x, y)$ are the vertices of an isosceles trapezium in which $AB \parallel CD$.
132. Prove that the straight lines $(a + b)x + (a - b)y - 2ab = 0, (a - b)x + (a + b)y - 2ab = 0$ and $x + y = 0$ form an isosceles triangle whose vertical angle is $2\tan^{-1}\frac{a}{b}$.
133. Find the angle between the lines $x = a$ and $by + c = 0$.
134. Find the tangent of the angle between the lines which have intercepts of $3, 4$ and $1, 9$ on $x$ and $y$ axes respectively.
135. Prove that the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} - \frac{y}{a} = 1$ are perpendicular to each other.
136. Show that the line joining $(2, -3)$ and $(-1, 2)$ is perpendicular to the line joining $(3, 7)$ and $(-2, 4)$.
137. A line passing through the points $(a, 2a)$ and $(-2, 3)$ is perpendicular to the line $4x + 3y + 5 = 0$; find the value of $a$.
138. Show that the lines $y = 7x + 2$ and $2y - 14x + 1 = 0$ are parallel.
139. Prove that the line $k^2x + ky + 1 = 0$ is perpendicular to the line $x - ky = 1$ for all real values of $k(k \neq)0$.
140. For what value of $k$ is the line $x - y + 2 + k(2x + 3y) = 0$ parallel to the line $3x + y = 0$.
141. Prove that the lines $2x - 3y + 1 = 0, x + y = 3, 2x - 3y = 2$ and $x = 4 - y$ form a parallelogram.
142. Find the value of $\theta$ between $0$ and $\pi$ if $x\cos\theta + y\sin\theta = 2$ is perpendicular to the line $x - y = 3$.
143. If the line $x - 3y + 5 + k(x + y - 3) = 0$, where $k$ is arbitrary, is perpendicular to the line $x + y = 1$, then find $k$ and the equation of the first line.
144. Prove that the median of an equilateral triangle is perpendicular to the corresponding side.
145. Prove that the diagonals of a rhombus are at right angles.
146. Find the equation of a line through $(3, 4)$ and parallel to the line $y = 3x + 5$.
147. Find the equation of the straight line through $(2, 3)$ and perpendicular to the line $4x - 3y = 10$.
148. Find the equation of the straight line which has $y$ intercept equal to $\frac{4}{3}$ and is perpendicular to the line $3x - 4y + 11 = 0$.
149. Find the equation of the perpendicular bisector of the line segment joining the points $(1, 2)$ and $(2, 3)$.

150. The line $x + y = a$ meets the axes of $x$ and $y$ at $A$ and $B$ respectively. A $\triangle AMN$ is inscribed in the $\triangle OAB$($O$ being the origin) with right angle at $N$. $M, N$ respectively lie on $OB$ and $AB$. If the area of the triangle $AMN$ is $\frac{3}{8}$th of the rea of the $\triangle OAB$, then find $AN : NB$.

151. Find the slope of the lines which make an angle of $45^\circ$ with the line $3x - y + 5 = 0$.

152. Find the equation of the lines through the point $(3, 2)$ which makes an angle of $45^\circ$ with the line $x - 2y = 3$.

153. A vertex of an equilateral triangle is $(2, 3)$ and the equation of the opposite side is $x + y = 2$. Find the equation of the other sides of the triangle.

154. A line $4x + y = 1$ through the point $A(2, -7)$ meets the line $BC$ whose equation is $3x - 4y + 1 = 0$ at the point $B$. Find the equation of the line $AC$, so that $AB = AC$.

155. Find the equation of straight lines passing through $(-2, -7)$ and having an intercept of length 3 between the straight lines $4x + 3y = 12$ and $4x + 3y = 3$.

156. Find the equation of the straight line parallel to $x + 2y = 3$ and passing through the point $(3, 4)$.

157. Find the equation of the straight line which passes through the point $(4, 3)$ and is parallel to the line $3x + 4y = 12$.

158. Find the equation of the straight line parallel to $3x - 4y + 6 = 0$ and passing through the middle point of the line segment made by $(2, 3)$ and $(4, -1)$.

159. Find the equation of the straight line passing through the point $(2, 1)$ and parallel to the line joining the points $(2, 3)$ and $(3, -1)$.

160. Find the equation of the straight line passing through the point $(\alpha, \beta)$ and parallel to the line $lx + my + n = 0$.

161. Find the equation of the straight line passing through the point $(2, 5)$ and perpendicular to the line $2x + 5y = 31$.

162. Find the equation to the straight line which passes through the point $(x', y')$ and is perpendicular to the line $yy' = 2a(x + x')$.

163. Find the angle between the straight lines $(m^2 - mn)y = (mn + n^2)x + n^3$ and $(mn + m^2)y = (mn - n^2)x + m^3$.

164. Prove that the equation of the straight line which passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and is perpendicular to the line $x\sec\theta + y\csc\theta = a$ is $x\cos\theta - y\sin\theta = a\cos 2\theta$.

165. Find the equation of the straight line through $(a\cos\theta, b\sin\theta)$ perpendicular to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$.

166. Two consecutive sides of a parallelogram are $4x+5y=0$ and $7x+2y=0$. If the equation of one of the diagonal is $11x+7y=9$, find the equation of the other diagnal.

167. Show that the area of the triangle whose sides are $y=m_1x+c_1, y=m_2x+c_2$ and $x=0$ is $\frac{1}{2}.\frac{(c_2-c_1)^2}{m_1.m_2}$.

168. Show that the area of the triangle formed by the lines $y=m_rx+c_r, r=1,2,3$ is $\frac{(c_1-c_2)^2}{2(m_1-m_2)}+\frac{(c_2-c_3)^2}{2(m_2-m_3)}+\frac{(c_3-c_1)^2}{2(m_3-m_1)}$.

169. Show that the area of the triangle whose sides are $a_rx+b_ry+c_r=0, r=1,2,3,$ is $\frac{\Delta^2}{2|C_1C_2C_3|}$, where $C_1, C_2$ and $C_3$ are the cofactors of $c_1, c_2$ and $c_3$ respectively in the determinant $\Delta=-\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

170. Show that the lines $4x+y-9=0, x-2y+3=0, 5x-y-6=0$ make equal intercepts on any line of gradient 2.

171. Find the coordinates of the foot of the perpendicular drawn from point $(2,3)$ to the line $y=3x+4$.

172. Find the image of the point $(-8,12)$ with respect to the line mirror $4x+7y+13=0$.

173. If the image of the point $(x_1,y_1)$ with respect to the mirror $ax+by+c=0$ be $(x_2,y_2)$, show that $\frac{x_2-x_1}{a}=\frac{y_2-y_1}{b}=-\frac{2(ax_1+by_1+c_1)}{a^2+b^2}$.

174. A ray of light is sent along the lines $x-2y-3=0$. Upon reaching the line $3x-2y-5=0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

175. A man starts from the point $P(-3,4)$ and will reach the point $Q(0,1)$ touching the line $2x+y=7$ at $R$. Find $R$ on the line sho that he will travel the shortest distance.

176. A ray of light is sent along the line $2x-3y=5$. After refracting across the line $x+y=1$ it enters the opposite side after turning by $15^\circ$ away from the line $x+y=1$. Find the equation of the line along which the refracted ray travels.

177. Find the equation of the straight line which passes through the point $(2,-2)$ and the point of intersection of the lines $5x-y=9$ and $x+6y=8$.

178. Find the equation of the straight line which passes through the point of intersection of the lines $x-y-1=0$ and $2x-3y+1=0$ and is parallel to the line $3x+4y=14$.

179. Find the equation for the straight line which passes through the point of intersection of the lines $3x-4y-7=0$ and $12x-5y-13=0$ and is perpendicular to the line $2x-3y+5=0$.

180. Find the equation of the straight lines passing through the point of intersection of the lines $x+3y+4=0$ and $3x+y+4-0$ and equally inclined to the axes.

181. The equation of two sides of a triangle are $3x - 2y + 6 = 0$ and $4x + 5y = 20$ and the orthocenter is $(1, 1)$. Find the equation of the third side.

182. Show that the diagonal of the parallelogram whose sides are $u = p, u = q, v = r, v = s$ where $u \equiv ax + by + c$ and $v \equiv a'x + b'y + c'$ and which passes through the points of intersection of lines $u = p, v = r$ and $u = q, v = s$ is given by $\begin{vmatrix} u & v & 1 \\ p & r & 1 \\ q & s & 1 \end{vmatrix} = 0$.

183. Show that the straight lines $x(a + 2b) + y(a + 3b) = a + b$ pass through a fixed point for different values of $a$ and $b$.

184. If $lx + my + n = 0$, where $l, m, n$ are variables is the equation of a variable line and $l, m, n$ are connected by the relation $al + bm + cn = 0$, where $a, b, c$ are constants, show that the line passes through a fixed point.

185. A variable line cuts $n$ given concurrent straight lines at $A_1, A_2, A_3, \ldots, A_n$, such that $\sum_{i=1}^{n} \frac{1}{OA_i}$ is a constant. Show that it always passes through a fixed point. $O$ is the point of intersection of the lines.

186. Prove that the straight lines $4x + 7y = 9, 5x - 8y + 15 = 0$ and $9x - y + 6 = 0$ are concurrent.

187. Prove analytically that medians of a triangle are concurrent.

188. Show that the lines $(p + q)x + (p + q)y - (p - q) = 0, (p - q)x - (p - q)y - (p + q) = 0, pq + qy - p = 0$ and $qx + py + q = 0$ are concurrent.

189. If the lines $p_1x + q_1y = 1, p_2x + q_2y = 1$ and $p_3x + q_3y = 1$ be concurrent, show that the points $(p_1, q_1), (p_2, q_2)$ and $(p_3, q_3)$ are collinear.

190. For what value of $m$, the line $mx + 2y + 5 = 0$ will pass through the point of intersection of the lines $x - 4y = 3$ and $x + 2y = 9$?

191. Find the point of intersection of the lines $yt_1 = x + at_1^2$ and $yt_2 = x + at_2^2$.

192. If the straight line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the point of intersection of the lines $x + y = 3$ and $2x - 3y = 1$ and is parallel to the line $y = x - 6$, find $a, b$.

193. Find the vertice and area of the triangle whose sides are $x = y, y = 2x$ and $y = 3x + 4$.

194. Find the area of the triangle which is formed by the lines $3x - 4y + 4a = 0, 2x - 3y + 4a = 0$ and $5x - y + a = 0$.

195. Show that the area of the triangle formed by the three straight lines $y_1 = m_1x, y = m_2x$ and $y = c$ is equal to $\frac{1}{4}c^2\sqrt{11}\left(\sqrt{3} + 1\right)$, where $m_1, m_2$ are the roots of the equation $x^2 + \left(\sqrt{3} + 2\right)x + \sqrt{3} - 1 = 0$.

196. Find the coordinates of the foot of the perpendicular drawn from the point $P(-8, 12)$ on the line $4x + 7y + 13 = 0$.

197. Find the projection of the point $(1, 0)$ on the line joining the points $P(-1, 2)$ and $Q(5, 4)$.

198. If perpendiculars are drawn from origin to the straight lines $x + 3y = 3$ and $2x + 3y = 5$, then find the equation of the line joining the foot of these perpendiculars.

199. If $(h, r)$ is the foots of the perpendiculars from $(x_1, y_1)$ to $lx + my + n = 0$ prove that $\frac{x_1-h}{l} = \frac{y_1-r}{m} = \frac{lx_1+my_1+n}{l^2+m^2}$.

200. Find the image of the point $(-8, 12)$ with respect to a line mirror $4x7y + 13 = 0$.

201. If the image of the point $(2, 1)$ with respect to a line mirror be $(5, 2)$, find the equation of the mirror.

202. Find the equation of the straight line which passes through the point $(1, 1)$ and the point of intersection of the lines $3x + 2y = 0$ and $x - 2y = 0$.

203. Find the equation of the straight line which passes through the point $(2, -2)$ and the point of the intersection of the lines $5x - y = 9$ and $x + 6y = 8$.

204. Find the equation of the straight line passing through the point of intersection of the lines $2x + y - 1 = 0$ and $x + 3y - 2 = 0$ and making with the coordinate axes a triangle of area $\frac{3}{8}$.

205. The sides $AB$ and $AD$ of a parallelogram $ABCD$ are $2x - y + 1 = 0$ and $x + 3y - 10 = 0$ respectively and $C$ is the point $(-1, -2)$. Find the equation of the diagonals $AC$ and $BD$.

206. Prove that the lines $2x - y - 5 = 0, 3x - y - 6 = 0$ and $4x - y - 7 = 0$ are concurrent.

207. Find the value of $m$ for which the two lines $mx + (2m + 3)y + m + 6 = 0$ and $(2m + 1)x + (m - 1)y + m - 9 = 0$ intersect at a point on $y$-axis.

208. Find the value of $m$ so that lines $y = x + 1, 2x + y = 16$ and $y = mx - 4$ may be concurrent.

209. If the three lines $ax + a^2y + 1 = 0, bx + b^2y + 1 = 0$ and $cx + c^2y + 1 = 0$ are concurrent, show that at least two of the three constants $a, b, c$ are equal.

210. Find the condition that the lines $y = m_1x + c_1, y = m_2x + c_2$ and $y = m_3x + c_2$ may be concurrent.

211. Show that the straight lines $(b + c)x + ay + 1 = 0, (c + a)x + by + 1 = 0$ and $(a + b)x + cy + 1 = 0$ are concurrent.

212. Prove analytically that the right bisectors of the sides of a triangle are concurrent.

213. Prove that perpendiculars drawn from the vertices to the opposite sides are concurrent.

214. Prove that the family of lines represented by $x(1 + \lambda) + y(2 - \lambda) + 5 = 0$, $\lambda$ beind arbitrary, pass through a fixed point. Also find the fixed point.

215. Prove that the line $x(a + 2b) + y(a - 3b) = a - b$ passes through a fixed point for different values of $a$ and $b$.

216. Find the centroid and incenter of the triangle whose sides are $3x - 4y = 0, 5x + 12y = 0$ and $y - 15 = 0$.

217. Find the coordinate of the orthocenter of the triangle whose vertices are $(0, 0), (2, -1)$ and $(-1, 3)$.

218. Find the coordinate of the orthocenter of the triangle whose sides are $3x - 2y = 6, 3x + 4y + 12 = 0$ and $3x - 8y + 12$.

219. Two vertices of a triangle are $(3, -1)$ and $(-2, 3)$ and its orthocenter is origin, find the coordinate of its third vertex.

220. A triangle has the lines $y = m_1x$ and $y = m_2x$ for two of its sides, where $m_1$ and $m_2$ are the roots of the equation $bx^2 + 2hx + a = 0$. If $H(a, b)$ is the orthocenter of the triangle, show that the equation of the third side is $(a + b)(ax + by) = ab(a + b - 2h)$.

221. A triangle is formed by the straight lines $ax + by + c = 0, lx + my + n = 0$ and $px + qy + r = 0$. Show that that straight line $\frac{px+qy+r}{ap+bq} = \frac{lx+my+n}{al+nb}$ passes through the orthocenter of the triangle.

222. The three sides of a triangle are $L_r \equiv x \cos\theta_r + y \sin\theta_r - p_r = 0, r = 1, 2, 3$. Show that the orthocenter of the triangle is given by $L_1 \cos(\theta_2 - \theta_3) = L_2 \cos(\theta_3 - \theta_1) = L_3 \cos(\theta_1 - \theta_2)$.

223. Find the centroid and incenter of the triangle whose sides have the equations $3x - 4y = 0, 12y + 5x = 0$ and $y - 15 = 0$.

224. The coordinates of the vertices $A, B$ and $C$ of the $\triangle\, ABC$ taken in anti-clockwise order are respectively $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$. Prove that the $\angle A$ is acute or obstuse according as $(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3) > 0$ or $< 0$. Also find the condition for the triangle to be right-angled at $A$.

225. Show that the four lines $4x - 3y = 5, x - 2y = 10, 7x + y = 40$ and $x + 3y + 10 = 0$ form the sides of a cyclic quadrilateral.

226. Find the condition for the quadrilateral to be cyclic whose sides are $a_rx + b_ry + c_r = 0; r = 1, 2, 3, 4$ taken in order.

227. Show that the lines $2x + 3y + 19 = 0$ and $9x + 6y - 17 = 0$ cut the coordinate axes in concyclic points.

228. Find the equation of the sides of a triangle having $B(-4, -5)$ as a vertex, $5x + 3y - 4 = 0$ and $3x + 8y + 13 = 0$ as the equation of two of altitudes not passing through $B$.

229. The straight line $L$ is perpendicular to the line $5x - y = 1$. The area of the triangle formed by $L$ and the coordinate axes is 5. Find the equation of the line.

230. The line $2x + 3y = 12$ meets the $x$-axis at $A$ and $y$-axis at $B$. The line through $(5, 5)$ perpendicular to $AB$ meets the $xy$ axes and $AB$ at $C, D, E$ respectively. If $O$ is origin of axes then find the area of $OCEB$.

231. A square has its center at origin and one vertex at $(1, 2)$. Find the equation of its sides.

232. $ABC$ is an equilateral triangle. $AD$ is its altitude through $A$. If $A \equiv (1, 2)$ and $D \equiv (-2, 6)$, find the equations of the sides of the triangle.

233. The equation of one side of an equilateral triangle is $x - y = 0$ and one vertex is $\left(2 + \sqrt{3}, 5\right)$. Prove that second side is $y + \left(2 - \sqrt{3}\right)x = 6$, and find the equation of the third side.

234. A diagonal of a square lies along the line $8x - 15y = 0$ and one vertex of the square is $(1, 2)$. Find the equations of the lines of the square passing through this vertex.

235. Find the equation of the lines which pass through $(4, 5)$ andd make equal angles with the lines $5y = 12x + 6$ and $3x = 4y + 7$.

236. Two equal sides of an isosceles triangle have the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third sides passes through the point $(1, -10)$. Determine the equation of the third side.

237. Prove that area of the triangle formed by the three straight lines $x \cos\alpha + y \sin\alpha - p_1 = 0, x \cos\beta + y \sin\beta - p_2$ and $x \cos\gamma + y \sin\gamma - p_3 = 0$ is $\frac{1}{2}\frac{\{p_1 \sin(\gamma-\beta)+p_2 \sin(\alpha-\gamma)+p_3 \sin(\beta-\alpha)\}^2}{|\sin(\gamma-\beta)\sin(\alpha-\gamma)\sin(\beta-\alpha)|}$.

238. Find the area of a triangle formed by the $y$-axis, the straight line $L$ passing through the points $(1, 1)$ and $(2, 0)$ and the straight line perpendicular to the line $L$ and passing through $\left(\frac{1}{2}, 0\right)$.

239. Find the coordinates of the feet of the perpendicular from the point $(9, 3)$ to the sides of the triangle whose vertices are at the points $(0, 0), (8, 0), (4, 8)$. Prove that the points so determined lies on a straight line and find its equation.

240. Obtain the coordinates $(\alpha, \beta)$ of the foot of the perpendicular from the origin to $\frac{x}{a} + \frac{y}{b} = 1$ and show that $(\alpha^2 + \beta^2)(\alpha + \beta) = (a + b)\alpha\beta$.

241. Find the equation of the diagonal through the origin of the quadrilateral formed by $x = 0, y = 0, x + y = 1, 6x + y = 3$.

242. The altitudes of a $\triangle\, ABC$ are respectively $AD, BE, CF$. If the points $A, D, E, F$ have the coordinates $(-4, 5), \left(\frac{16}{5}, -\frac{23}{5}\right), (4, 1), (-1, -4)$, find the coordinates of other vertices of the triangle.

243. Prove that the lines $y = m_r x + c_r; r = 1, 2, 3$ cut off equal intercepts on th etransversal $x + y = 1$ if $1 + m_1, 1 + m_2, 1 + m_3$ are in H.P.

244. A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point $(1, 5)$. Obtain its equation.

245. If the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ cut the coordinate axes in cyclic points, prove that $|a_1 a_2| = |b_1 b_2|$.

246. A rectangle $ABCD$ is inscribed in a circle with a diameter lying along the line $3y = x + 10$. If $A$ and $B$ are the points $(-6, 7)$ and $(4, 7)$ respectively, then find the area of the rectangle.

247. From the point $(2,5)$ rays of light are sent at $45^{\circ}$ with the line $2x+y=1$. Find the equation of the lines of the reflected rays if the rays reflect from $x+2y=1$.

248. A ray of light is sent along the straight line $y=\frac{2x}{3}-4$. On reaching the $x$-axis it is reflected. Find the point of incidence and the equation of the reflected ray.

249. From the point $M(-2,3)$ a ray of light is sent at an angle $\alpha$ to the positive direction of $x$-axis. Upon reaching the $x$-axis the ray is reflected from it. Find the equation of the reflected ray if $\tan\alpha=3$.

250. A light beam emanating from the point $(3,10)$ reflects from the straight line $2x+y-6=0$ and then passes through the point $B(7,2)$. Find the equations of the incident and reflected beams.

251. The line $3x+2y=24$ meets $y$-axis at $A$ and $x$-axis at $B$. The perpendicular bisector of $AB$ meets the line through $(0,-1)$ parallel to $x$-axis at $C$. Find the area of the $\triangle\, ABC$.

252. Find the condition that the real line $ax+by+c=0, bx+cy+a=0$ and $cx+ay+b=0$ are concurrent.

253. Find the condition that the lines $(b-c)x+(c-a)y+a-b=0, (c-a)x+(a-b)y+b-c=0$ and $(a-b)x+(b-c)x+c-a=0$.

254. Prove that the determinant

$$\begin{vmatrix} x_2-x_3 & y_2-y_3 & x_1(x_2-x_3)+y_1(y_2-y_3) \\ x_3-x_1 & y_3-y_2 & x_2(x_3-x_1)+y_2(y_3-y_1) \\ x_1-x_2 & y_2-y_2 & x_3(x_1-x_2)+y_3(y_1-y_2) \end{vmatrix}=0.$$

What geometrical property does it imply for a triangle whose vertices are $(x_r,y_r); r=1,2,3$?

255. Prove that all lines represented by the equation $(2\cos\theta+3\sin\theta)x+(3\cos\theta-5\sin\theta)y-(5\cos\theta-2\sin\theta)=0$ pass through a fixed point for all values of thete. Find the coordinates of that point and its reflection in the line $x+y=\sqrt{2}$.

256. Prove that the orthocenter of the triangle formed by the three lines $y=m_1x+alm_1, y=m_2x+alm_2, y=m_3x+alm_3$ is $\left\{-a,\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_1m_2m_3}\right)\right\}$.

257. If the coordinates of the point $P,Q,R$ satisfy the relation $xy=c^2$, show that the orthocenter of $\triangle\, PQR$ also satisfies the relation.

258. $A$ and $B$ are two fixed points $(3,2)$ and $(5,1)$ respectively. $\triangle\, ABP$ is equilateral and is situated on the side of $AB$ remote from the origin. Find the coordinates of $P$ and the orthocenter of the $\triangle\, ABP$.

259. Vertices of a triangle are $A(x_1,x_1\tan\alpha_1), B(x_2,x_2\tan\alpha_2)$ and $C(x_3,x_3\tan\alpha_3)$. If the circumcenter coincide with the origin and the orthocenter $H$ is $(\overline{x},\overline{y})$ then prove that $\overline{y}\left(\cos\alpha_1+\cos_{\alpha_2}+\cos\alpha_3\right)=\overline{x}(\sin\alpha_1+\sin\alpha_2+\sin\alpha_3)$, where $x_1\sec\alpha_1, x_2\sec\alpha_2, x_3\sec\alpha_3$ have the same sign.

260. Find the area and the orthocenter of the triangle formed by the lines $x+ly=l^2, x+my=m^2$ and $l+ny=n^2$.

261. If the equation of the sides of a triangle are respectively $a_1x + b_1y = 1, a_2x + b_2y = 1$ and $a_3x + b_3y = 1$ and whose orthocenter is the origin, prove that $a_1a_2 + b_1b_2 = a_2a_3 + b_2b_3 = a_3a_1 + b_3b_1$.

262. Prove that the $\triangle\, DEF$ has the same centroid as $\triangle\, ABC$, where $D, E, F$ are the middle points of the sides of the later triangle. Also prove that the orthocenter of the $\triangle\, DEF$ coincides with the circumcenter of the $\triangle\, ABC$.

263. The circumcenter of a triangle with vertices $A(a, \tan\alpha), B(b, \tan\beta)$ and $C(c, \tan\gamma)$ lies at the origin $\alpha + \beta + \gamma = \pi$, show that its orthocenter lies on the line $4\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\right)x - 4y\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = y$.

264. Show that the line $\frac{a_1x+b_1y+c_1}{a_1a_3+b_1b_3} = \frac{a_2x+b_2y+c_2}{a_2a_3+b_2b_3}$ passes through the orthocenter of the triangle formed by the lines $a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$.

265. Find the position of the points $(1, 1)$ and $(2, -1)$ with respect to the line $3x + 4y - 6 = 0$.

266. Show that the four points $(0, 0), (-1, 1), (-7, -4)$ and $(9, 6)$ are in the four different compartments made by the two straight lines $2x - 3y + 1 = 0$ and $3x - 5y + 2 = 0$.

267. Find the position of the origin w.r.t. the triangle whose sides are $x + 1 = 0, 3x - 4y = 5, 5x + 12y = 27$.

268. Show that the line segment joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is cut by the line $ax + by + c = 0$ in the ratio $-\frac{ax_1+by_1+c}{ax_2+by_2+c}$. Explain the minus sign.

269. A line $L$ intersects three sides $BC, CA$ and $AB$ of a $\triangle\, ABC$ in $P, Q$ and $R$ respectively. Prove that $BP.CQ.AR + PC.QA.BR = 0$.

270. Derive the condition to be imposed on $\beta$ so that $(0, \beta)$ should lie on or inside the triangle having sides $y + 3x + 2 = 0, 3y - 2x = 5$ and $x + 4y = 14$.

271. A rhombus has two consecutive vertice at $(2, 3)$ and $(-2, 6)$ and two of the sides are parallel to $2x + y = 1$. Find the other vertices of the rhombus if $(0, 3)$ is an interior point of rhombus.

272. Examine wheher the points $(3, -4)$ and $(2, 6)$ are on the same side or opposite sides of the line $3x - 4y = 8$?

273. Prove that the points $(2, -1)$ and $(1, 1)$ are on the opposite sides of the straight line $3x + 4y - 6 = 0$.

274. Find the position of the points $(3, 4)$ and $(-1, 1)$ w.r.t the line $6x + y - 1 = 0$.

275. Prove that the points of intersection of the line $x - y = 2$ with the parallel lines $2x + y = 7$ and $2x + y = 16$ are on the opposite sides of the line $x + y = 5$.

276. Find the distance of the point $(4, 5)$ from the straight line $3x - 5y + 7 = 0$.

277. Find the distance of the point $(1, 2)$ from the straight line with slope 5 and passing through the point of intersection of $x + 2y = 5$ and $x - 3y = 7$.

278. The equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is $(2, -1)$. Find the length of the side of the triangle.

279. If $a$ and $b$ are the intercepts of a straight line on the $x$ and $y$ axes respectively and $p$ be its perpendicular distance from the origin, prove that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

280. If $p$ and $p'$ be the lengths of perpendiculars from origin to the lines $x \sec\theta - y \csc\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$ respectively, show that $4p^2 + p'^2 = a^2$.

281. Find the equation of straight line which cuts off intercepts on $x$-axis twice that on $y$-axis and is at a unit distance from the origin.

282. Find the distance between the parallel lines $ax + by + c = 0$ and $ax + by + d = 0$.

283. Prove that the length of the perpendiculars from points $(m^2, 2m), (mn, m + n)$ and $(n^2, 2n)$ to the line $x\cos\theta + y\sin\theta + \frac{\sin^2\theta}{\cos\theta}$ form a G.P.

284. A straight road passes through two towns, one 5 km east and the other $2\frac{1}{2}$ km north from a tower. Where should a rest house be constructed by the side of the road so that it may be nearest to the tower.

285. A straight line is such that the algebraic sum of the perpendiculars upon it from any number of fixed points is zero. Show that the line always passes through a fixed point.

286. The coordinates of the extremeties $A$ and $B$ of a rod are $(1, 2)$ and $(3, 4)$ respectively. $S(0, 0)$ is a point source of light. The rod $AB$ is parallel to the wall and is at equal distance from $S$ and the wall. If $CD$ is the shadow of $AB$ on the wall, find the coordinates of $C$ and $D$ and the lnegth $CD$. $S, AB$ and $CD$ are planar.

287. Prove that the diagonals of the parallelogram formed by the lines $\frac{x}{a} + \frac{y}{b} = 1, \frac{x}{b} + \frac{y}{a} = 1, \frac{x}{a} + \frac{y}{b} = 2$ and $\frac{x}{b} + \frac{y}{a} = 2$ are at right angles.

288. Find the area of the parallelogram whose sides are $y = mx + a, y = mx + b, y = nx + c$ and $y = nx + d$.

289. Find the distance of the point of intersection of the lines $2x + 3y = 21$ and $3x - 4y + 11 = 0$ from the line $8x + 6y + 5 = 0$.

290. Find the length of the perpendicular drawn from the origin upon line joining the points $(a, b)$ and $(b, a)$.

291. Find the length of the perpendicular from the point $(4, -7)$ to the line joining the origin and point of intersection of the lines $2x - 3y + 14 = 0$ and $5x + 4y - 7 = 0$.

292. Find the equation of two straight lines which are parallel to $x + 7y + 2 = 0$, and at unit distance from the point $(1, -1)$.

293. Find the equations of the two straight lines parallel to $3x - 4y = 5$ at a unit distance from it.

294. Find the equation of two lines through $(0, a)$ which are at a distance $a$ from the point $(2a, 2a)$.

295. Find the equation of the line through the point of intersection of the lines $x-3y+1=0$ and $2x+5y-9=0$ and whose distance from origin is $\sqrt{5}$.

296. Find the equation of the straight line passing the point of intersection of the lines $x-y+1=0$ and $2x-3y+5=0$ and at a distance $\frac{7}{5}$ from the point $(3,2)$.

297. If the length of the perpendicular from the point $(1,1)$ to the line $ax-by+c=0$ be 1, show that $\frac{1}{c}+\frac{1}{a}-\frac{1}{b}=\frac{c}{2ab}$.

298. Show that the product of the perpendiculars on the line $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ from the points $\left(\pm\sqrt{a^2-b^2},0\right)$ is $b^2$.

299. Prove that the perpendicular distance between the lines $4x+3y=11$ is $8x+6y=15$ is $\frac{7}{10}$.

300. Prove that the lines $2x+3y=19$ and $2x+3y+7=0$ are equidistant from the line $2x+3y=6$.

301. Find the distance between the lines $y=mx+c$ and $y=mx+c_1$.

302. The equation of two sides of a square whose area is 25 square units are $3x-4y=0$ and $4x+3y=0$. Find the equation of the other two sides of the square.

303. Show that the parallelogram formed by $ax+by+c=0, a_1x+b_1y+c=0, ax+by+c_1=0$ and $a_1x+b_1y+c_1=0$ will be a rhombus if $a^2+b^2=a_1^2+b_1^2$.

304. For the straight lines $4x+3y-6=0$ and $5x+12y+9=0$, find the equation of the

    a. bisector of the obtuse angle between them,

    b. bisector of the acute angle between them, and

    c. bisector of the angle which contains the origin.

305. Prove that the length of the perpendiculars drawn from any point of the line $7x-9y+10=0$ to the lines $3x+4y-5=0$ and $12x+5y=7$ are equal.

306. Prove that the internal bisectors of the angle of a triangle meet in a given point.

307. Find the coordinates of the incenter of the triangle whose sides are $x+1=0$, $3x-4y-5=0$, and $5x+12y-27=0$.

308. Two opposite sides of a rhombus are $x+y=1$ and $x+y=5$. If one vertex is $(2,-1)$ and the angle at that vertex be $45^{\circ}$, find the vertex opposite to given vertex.

309. Two sides of a rhombus $ABCD$ are parallel to the lines $y=x+2$ and $y=7x+3$. If the diagonals of the rhombus intersect at the point $(1,2)$ and the vertex $A$ is on $y$-axis, find the possible coordinates of $A$.

310. Find the equations of the bisectors of the angle between the lines $x-2y+3=0$ and $4x+2y-5=0$.

311. Prove that the line $6x+66y-7=0$ is a bisector of the angle between the lines $15x-18y-1=0$ and $12x+10y-3=0$.

312. Show that each point on the line $2x + 11y = 5$ is at equal distance from the lines $24x + 7y = 20$ and $4x - 3y = 2$.
313. Find the locus of the point equidistant from the lines $6x + 8y - 10 = 0$ and $4x - 3y = 7$.
314. Find the equartions of the bisectors of the angles between the lines $x + y - 3 = 0$ and $7x - y + 5 = 0$ and state which of them bisects the acute angle between the lines.
315. Prove that the bisector of the acute angle between the lines $3x + 4y = 11$ and $12x - 5y = 2$ is $11x + 3y = 17$.
316. Find the equation of the line which bisects the obtuse angle between the lines $x - 2y + 4 = 0$ and $4x - 3y + 2 = 0$.
317. Show that the four points $(1, 0), (2, 3), (1, -4)$ and $(8, 1)$ lie in the four compartments made by the lines $x + y - 2 = 0$ and $x - y - 3 = 0$.
318. Determine whether the origin lies inside or outside the triangle whose saides are given by the equation $7x - 5y - 11 = 0, 8x + 3y + 31 = 0$, and $x + 8y - 19 = 0$.
319. Sides of a square lie on the line $5x - 12y - 65 = 0$ and $5x - 12y + 26 = 0$. Find the area of the square.
320. The equation of two sides of a square are $3x + 4y - 5 = 0$ and $3x + 4y - 15 = 0$. The third side has a point $(6, 5)$ on it. Find the equations of this and the remaining side of the square.
321. The equation of one side of rectangle is $3x - 4y - 10 = 0$ and the coordinates of two of its vertices are $(2 -, 1)$ and $(2, 4)$. Find the area of rectangle and the equation of that diagonal of the rectangle which passes through the point $(2, 4)$.
322. Prove that the lines $ax \pm by \pm c = 0$ enclose a rhombus whose area is $\frac{2x^2}{|ab|}$.
323. If $p, q, r$ be the lengths of perpendiculars from the vertices $A, B, C$ respectively of the $\triangle ABC$ on any straight line, then prove that $a^2(p - q)(p - r) + b^2(q - r)(q - p) + c^2(r - p)(r - q) = 4\Delta^2$, where $\Delta$ is the area of the triangle and $a, b, c$ length of the sides opposite to angles $A, B, C$ respectively.
324. Prove that no line can be drawn through the point $(4, -5)$ so that its distance from $(-2, 3)$ will be equal to 12.
325. The vertices of $\triangle OBC$ are $O(0, 0), B(-3, -1)$ and $C(-1, -3)$. Find the equation of the line parallel of $BC$ and intersecting with $OB$ and $OC$, whose perpendiculars distance from $O$ is $\frac{1}{2}$.
326. The point $(1, -1)$ is the center of a square, one of whose sides lies on the line $x - 2y + 12 = 0$. Find the equation of the straight lines which contain the remaining sides of the square.
327. The equation of two sides of a parallelogram are $3x - 2y + 12 = 0$ and $x - 3y + 11 = 0$ and the point of intersection of its diagomals is $(2, 2)$. Find the equation of other two sides and its diagonals.

328. Given three parallel lines $3x + 4y + 2 = 0, 3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$. Show that the first of them lies between the other two. Also find the ratio in which the line divides the distance between the other two.

329. The three lines $x + 2y + 3 = 0, x + 2y - 7 = 0, 2x - y - 4 = 0$ form the three sides of two squares. Find the equation of the fourth side of each square.

330. Find the equation of the internal bisectors of the angled of the triangle whose sides are $3x + 4y = 6, 12x - 5y = 3$ and $4x - 3y + 12 = 0$.

331. Find the incenter of the triangle whose sides are $3x + 4y - 12 = 0, 5x + 12y - 20 = 0$ and $24y - 7x - 22 = 0$.

332. Show that the reflection of the line $px + qy + r = 0$ in the line $x + y + 1 = 0$ is the line $qx + py + (p + q - r) = 0$, where $p \neq -q$.

333. A man at the corssing of two roads $x - 2y - 4 = 0$ and $2x - y - 4 = 0$ starts walking along the bisector of the acute angle between the roads and after covering a distance of 2 km reaches the bank of a straight river at right angle to its path. Find the equation of the bank and the coordinates of the point where the path meets the bank.

334. A rhombus has two of its sides parallel to the lines $y = 2x + 3$ and $y = 7x + 2$. If the diagonals cut at $(1, 2)$ and one vertex is on the $y$-axis find the possible coordinates of that vertex.

335. A straight line segment of length $l$ moves with its end on two mutually perpendicular lines. Find the locus of the poitn which divides the segment in the ratio $1 : 2$.

336. Find the locus of the middle point of the potion of the line $x \cos\alpha + y \sin\alpha = p$ which is intercepted between the axes given that $p$ remains constant.

337. A variable straight line, drawn through the point of intersection of the straight line $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, meets the axes at $A$ and $B$. Show that the locus of the mid-point of $AB$ is the curve $2xy(a + b) = ab(x + y)$.

338. If the line $\frac{x}{a} + \frac{y}{b} = 1$ moves in such a way that $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$, where $c$ is constant, prove that the foot of perpendicular from the origin on the straight line described the circle $x^2 + y^2 = c^2$.

339. A straight line passes through a fixed point $(h, k)$; find the locus of the foot of the perpendicular on it drawn from the origin.

340. $OX$ and $OY$ are two straight lines at right angles to one another. On $OY$ a fixed point $A$ is taken and on $OX$ any point $B$. On $AB$ an equilateral triangle is described, its vertex $C$ being on the side $AB$ away from $O$. Show that the locus of the triangle is a straight line.

341. A point $P$ is such that its perpendicular distance from the line $y - 2x + 1 = 0$ is equal to its distance from the origin. Find the equation of the locus of the point $P$. Prove that the line $y = 2x$ meets the locus in two points $Q$ and $R$ such that the origin is the mid-point of $QR$.

342. A line drawn through the origin intersect the lines $2x + y - 2 = 0$ and $x - 2y + 2 = 0$ in $A$ and $B$. Find the locus of the mid-point of segment $AB$.

343. $O$ is a fixed point and $AP$ and $BQ$ are two fixed parallel straight lines. $BOA$ is perpendicular to both and $\angle POQ$ is a right angle. Prove that the locus of the foot of the perpendicular from $O$ on $PQ$ is a circle, whose diameter is $AB$.

344. Two fixed points $P$ and $Q$ are given. $R$ is a variable point on one side of the line $PQ$ such that $\angle RPQ - \angle RQP$ is a positive constant $2\alpha$. Find the locus of point $R$.

345. A variable straight line passes through the points of intersection of the lines $x + 2y = 1$ and $2x - y = 1$ and meets the coordinate axes at $A$ and $B$. Find the locus of middle point of $AB$.

346. Let $L_1 = 0$ and $L_2 = 0$ be two fixed lines. A variable line is drawn through the origin to cut the two lines at $A$ and $B$. $P$ is a point on the line $AB$ such that $\frac{m+n}{OP} = \frac{m}{OR} + \frac{n}{OS}$. Show that the locus of $P$ is a straight line through the point of intersection of the given lines $(R, S, P)$ are on the same side of origin.

347. Given $n$ straight lines and a fixed point $O$. Through $O$ a straight line is drawn meeting these lines at the points $R_1, R_2, ..., R_n$ and a point $R$ is taken on it such that $\frac{n}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} + \cdots + \frac{1}{OR_n}$. Show that the locus of $R$ is a straight line.

348. The base of a triangle passes through a fixed point $(f, g)$ and its sides are respectively bisected at right angles by the lines $y^2 - 8xy - 9x^2 = 0$. Determine the locus of its vertex.

349. Having given the bases and the sum of the areas of a number of triangles is constant, which have a common vertex, show that the locus of this vertex is a straight line.

350. If $A(\cos t, \sin t), B(\sin t, -\cos t), C(1, 2)$ are the vertices of a $\triangle\, ABC$, find the locus of centroid if $t$ varies.

351. The position of a moving point in the $xy$-plane given at a time $t$ is given by $u \cos\alpha, u \sin\alpha - \frac{1}{2}gt^2$, where $u, \alpha, g$ are constants. Find the locus of the moving points.

352. A straight line passing through the point $(1, 1)$ is terminated by the axes of coordinates. Show that the locus of the mid-point of the line has the equation $2xy = x + y$.

353. Find the locus of the middle point of the intercepts made by the axes on the lines drawn through the point $(\alpha, \beta)$.

354. A straight line moves such that the sum of its intercepts on the axes is $k$. Find the locus of the middle point of the portion of the line intercepted between the axes.

355. A line $APB$ of constant length meets the $x$-axis at $A$ and $y$-axis at $B$. If $AP = b, PB = a$ and the line slides with its extremities on the coordinate axes, show that the equation of the locus of the point $P$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

356. A variable line through the point $\left(\frac{6}{5}, \frac{6}{5}\right)$ cuts the coordinate axes at points $A$ and $B$. If the point $P$ divides $AB$ internally in the ratio $2:1$, show that the equation of the locus of $P$ is $5xy = 2(2x + y)$.

357. A straight line moves in such a way that the length of the perpendicular upon it from the origin is always $p$. Find the locus of the centroid of the triangle which is formed by the line and the axes.

358. Two fixed points $A$ and $B$ have coordinates $(x_1, y_1)$ and $(x_2, y_2)$. A point $P$ moves such that $AP$ is perpendicular to $BP$. Show that the locus of $P$ is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

359. A point moves so that the square of its distance from the point $(3, -2)$ is numerically equal to its distance from the line $5x - 12y = 13$. Find the equation of its locus.

360. A point moves such that the sum of its distance from two fixed points $(ae, 0)$ and $(-ae, 0)$ is always $2a$. Prove that its locud is $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$.

361. Find the locus of the middle point of the intercept on the line $y = x + c$ made by the line $2x + 3y = 5$ and $2x + 3y = 8$, $c$ being a parameter.

362. If a line $AB$ of length $2l$ moves with end $A$ always on $x$-axis and the end $B$ always on the line $y = 6x$. Find the equation of the locus of the mid-point of $AB$.

363. $P$ is the point $(-1, 2)$. A variable line through $P$ cuts the coordinate axes at $A$ and $B$ respectively. $Q$ is a point on $AB$ such that $PA, PQ, PB$ are in H.P. Show that the locus of $Q$ is the line $y = 2x$. ($A, B$ lie on the same side of $P$) Also show that in general locus of $Q$ is the rhombus whose sides are $y = 2x, y = -2x + 4, y = -2x - 4$ and $y = 2x + 8$ excluding the vertices.

364. If $O$ is the origin, $A$ is the point $(4, 4)$, $B$ is any point on the plane, find the locus of the point of intersection of the perpendicular bisectors of $OB$ and $AB$.

365. Two fixed points $A$ and $B$ are taken on the two axes such that $OA = a$ and $OB = b$. Two variable points $C$ and $D$ are taken on the same axes respectively, find the locus of the point of intersection of $AD$ and $BC$ if $\frac{1}{OC} - \frac{1}{OD} = \frac{1}{OA} - \frac{1}{OB}$.

366. $Q$ is any point on the line $x = a$. If $A$ is the fixed point $(a, 0)$ and $QR$, the bisector of the angle $OQA$, meets the $x$-axis in $R$, find the locus of the foot of the perpendicular from $R$ to $OQ$.

367. A right-angled $\triangle ABC$ having a right angle at $C, CA = b$ and $CB = a$ moves such that the angular points $A$ and $B$ slide along $x$-axis and $y$-axis respectively. Find the locus of $C$.

368. Show that the locus of a point which moves such that the square of its distance from the base of an isosceles triangle is equal to the rectangle under its distance from the other sides, is a circle.

369. A variable straight line is drawn through a given point $O$ to cut two fixed straight lines in $R$ and $S$; on it is taken a point $P$ such that $\frac{2}{OP} = \frac{1}{OR} + \frac{1}{OS}$, show that the locus of $P$ is a third fixed straight line.

370. If $p, x_1, x_2, ..., x_i, ...$ and $q, y_1, y_2, ..., y_i, ...$ from two infinite arithmetic sequences with common difference $a$ and $b$ respectively then find the locus of the point $(\alpha, \beta)$ where $\alpha = \frac{1}{n} \sum_{\{i=1\}}^{n} x_i$ and $\beta = \frac{1}{n} \sum_{\{i=1\}}^{n} y_i$.

# 4 Pair of Straight Lines

Consider a pair of straight lines $ax + by + c = 0$ and $a_1x + b_1y + c_1 = 0$, which we represent as $(ax + by + c)(a_1x + b_1y + c_1) = 0$.

Let $P(\alpha, \beta)$ on the line $ax + by + c = 0 \Rightarrow a\alpha + b\beta + c = 0$ and $(a\alpha + b\beta + c)(a_1\alpha + b_1\beta + c_1) = 0$.

Thus, a point which lies on either line wil satisfy $(ax + by + c)(a_1x + b_1y + c_1) = 0$. This equation represents a pair of straight lines.

## 4.1 Homogeneous Equations

Any equation in which combined powers of $x$ and $y$ is constant(say $n$) is called a homogeneous equation of degree $n$. For example, $ax^2 + 2hxy + by^2 = 0$ is a homogeneous equation of degree 2.

## 4.2 Pair of Straight Lines Through Origin

We will show that any homogeneous equation of second degree in $x$ and $y$ represents a pair of straight lines through the origin.

Consider the equation $ax^2 + 2hxy + by^2 = 0 \Rightarrow a + b\left(\frac{y}{x}\right)^2 + 2h\frac{y}{x} = 0$, which is a quadratic equation in $\frac{y}{x}$.

Let the roots be $m_1$ and $m_2$ of the above equation. $\Rightarrow \frac{y}{x} = m_1, m_2 \Rightarrow b(y - m_1x)(y - m_2x) = 0$

Thus, the given homogeneous equation represents two straight lines through origin.

If the lines are represented by $ax^2 + 2hxy + by^2 = 0$ and $y - m_1x = 0$ and $y - m_2x = 0$, then

$$m_1 + m_2 = -\frac{2h}{b} \tag{4.1}$$

and

$$m_1m_2 = \frac{a}{b} \tag{4.2}$$

.

## 4.3 Angle Between the Lines Represented by $ax^2 + 2hxy + by^2 = 0$

The tangent of the angle between the two lines is given by $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2}$

$= \pm\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2} = \pm\frac{2\sqrt{h^2-ab}}{a+b}$.

For lines to be perpendicular

$$a + b = 0 \tag{4.3}$$

and for them to be parallel

$$h^2 = ab \tag{4.4}$$

.

## 4.4 Bisectors of Angles Between the Pair of Straight Lines

Coninuing from previous sections the equation of bisectors of straight lines is given by $\frac{y-m_1x}{\sqrt{1+m_1^2}} = \pm\frac{y-m_2x}{\sqrt{1+m_2^2}}$

Combined equation is given by

$\left(\frac{y-m_1x}{\sqrt{1+m_1^2}} - \frac{y-m_2x}{\sqrt{1+m_2^2}}\right)\left(\frac{y-m_1x}{\sqrt{1+m_1^2}} + \frac{y-m_2x}{\sqrt{1+m_2^2}}\right) = 0$

$\Rightarrow \frac{(y-m_1x)^2}{1+m_1^2} - \frac{(y-m_2x)^2}{1+m_2^2} = (m_1 + m_2)(x^2 - y^2) + 2(m_1m_2 - 1)xy = 0$

$\Rightarrow -\frac{2h}{b(x^2-y^2)} + 2(\frac{a}{b} - 1)xy = 0 \Rightarrow \frac{x^2-y^2}{a-b} = \frac{xy}{h}$.

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \tag{4.5}$$

## 4.5 Condition for General Equation of Second Degree to Represent a Pair of Straight Lines

General equation of the second degree is given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. This equation will result into two straight lines if it can be resolved into two linear factors.

Let the straight lines be $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$, which are represented by the above equation.

Then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$

Comparing coefficients $l_1l_2 = a, m_1m_2 = b, n_1n_2 = c, m_1n_2 + m_2n_1 = 2f, n_1l_2 + n_2l_1 = 2g, l_1m_2 + l_2m_1 = 3h$

Multiplying last three we obtain

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \tag{4.6}$$

, which is the required condition.

The above condition can be represented as

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \tag{4.7}$$

The general equation can be treated as an equation in $x$ and then discriminat must be a perfect square for it to resolve into linear factors. We will obtain the same condition as above using this method as well.

## 4.6 Problems

1. Find the joint equation of the straight lines represented by $x = 4y - 2$ and $x - 1 = y$.
2. Prove that the equations to the straight lines passing through the origin and making an angle $\alpha$ with the straight line $y + x = 0$ are given by $x^2 + 2xy \sec 2\alpha + y^2 = 0$.
3. A pair of perpendicular straight lines are drawn through the origin forming with the line $2x + 3y = 6$ an isosceles triangle at the origin. Find the equation of pair of straight lines and the area of the triangle.
4. Find the combined equation of the lines $2x - y = 3$ and $y = 3x + 4$.
5. Find the joint equation of the lines through $(1, 2)$ and parallel to the lines $x - 2y = 5$ and $x = 3y - 4$.
6. Find the combines equation of the lines bisecting the angles between $x$ and $y$ axes.
7. Prove that the equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents two straight lines.
8. If the equation $6x^2 + 2kxy + 12y^2 + 22x + 31y + 20 = 0$ represent a pair of straight lines, find the value of $k$.
9. Show that the equation $10x^2 - 11xy - 6y^2 - 12x - y + 2 = 0$ represents a pair of straight lines.
10. Does the equation $2x^2 - 15xy - 17y^2 + 4x + 23y - 6 = 0$ represent a pair of straight lines?
11. For what value of $m$ does the equation $mx^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ represent two straight lines? Prove that they are perpendicular to each other.
12. For what values of $m$ does the equation $x^2 + mxy - 2y^2 + 3y - 1 = 0$ represent two straight lines?
13. If the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + m = 0$ represent two straight lines, find the value of $m$.
14. For what value of $m$ does the equation $6x^2 + 5xy - 4y^2 + 7x + my + 2 = 0$ represent two straight lines?
15. Find the angle between the pair of straight lines represented by the equation $4x^2 + 24xy + 11y^2 = 0$.
16. Find the angle between the pair of straight lines represented by the equation $(x^2 + y^2) \sin^2 \alpha = (x \cos \beta - y \sin \beta)^2$.
17. Find the angle between the pair of straight lines represented by the equation $x^2 - 5xy + 4y^2 + 3x - 4 = 0$.
18. Show that the two straight lines $x^2(\tan^2 \theta + \cos^2 \theta) - 2xy \tan \theta + y^2 \sin^2 \theta = 0$ make with the axis of $x$ angles such that the difference of their tangents is 2.
19. Find the length of the straight line joining the foot of perpendicular from the point $(p, q)$ on the pair of lines $ax^2 + 2hxy + by^2 = 0$.

20. A point moves so that the distance between the feet of the perpendiculars drawn from it to the lines $ax^2 + 2hxy + by^2 = 0$ is a constant $k$. Show that the equation of its locus is $(x^2 + y^2)(h^2 - ab) = k^2[(a-b)^2 + 4h^2]$.
21. Find the angle between the pair of straight lines given by $x^2 - 3xy - y^2 = 0$.
22. Find the angle between the lines given by $x^2 + 2xy \cot 2\alpha - y^2 = 0$.
23. Find the angle between the lines given by $x^2 - 2pxy + y^2 = 0$.
24. Show that the two straight lines given by $x^2 - 2xy \sec\theta + y^2 = 0$ make an angle $\theta$ with one another.
25. Find the angle between the lines represented by $(x^2 + y^2)\sin^2\alpha = (x\cos\alpha - y\sin\alpha)^2$.
26. Prove that the equation $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$ represents twp straight lines, which are perpendicular to each other.
27. Prove that the equation $16x^2 + 24xy + 9y^2 + 40x + 30y - 75 = 0$ represents two parallel straight lines.
28. Prove that the equation $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$ represents two straight lines. Also, find the angle between them.
29. If the equation $12x^2 + 7xy - py^2 - 18x + qy + 6 = 0$ represents two perpendicular straight lines, find the values of $p$ and $q$.
30. Prove that the equation $2x^2 + 3xy - 2y^2 = 0$ represents two lines through origin which are perpendicular to each other.
31. Find the separate equation of the lines represented by $2x^2 - xy - y^2 + 9x - 3y + 10 = 0$.
32. Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the coordinates of their point of intersection and also the angle between them.
33. Show that the equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents a pair of parallel straight lines. Also, find the perpendicular distance between them.
34. Find the combined equation of the straight lines passing through the point $(1, 1)$ and parallel to the lines represented by the equation $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$ and find the angle between them.
35. If the lines represented by $2x^2 - 5xy + 2y^2 = 0$ be the two sides of a parallelogram and the line $5x + 2y = 1$ be one of its diagonals, find the equation of the other diagonal and area of the parallelogram.
36. The base of a triangle passes through a fixed point $(f, g)$ and its sides are bisected at right angle by a pair of straight lines $y^2 - 8xy - 9x^2 = 0$. Determine the locus of its vertex.
37. Prove that the straight lines represented by $(y - mx)^2 = a^2(1 + m^2)$ and $(y - nx)^2 = a^2(1 + n^2)$ form a rhombus.

38. If the equation $2hxy + 2gx + 2fy + c = 0$ represents two straight lines, show that they form a rectangle of area $\frac{|fg|}{h^2}$ with the coordinate axes.
39. Find separate equations of lines represented by $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$.
40. Prove that the lines represented by $x^2 - 6xy + 3y^2 = 0$ are perpendicular to the lines represented by $3x^2 + 6xy + y^2 = 0$.
41. Show that one of the lines given by $y^2 + xy - 12x^2 = 0$ coincides with one of the lines given by $4y^2 - 13xy + 3x^2 = 0$ and the other two lines are perpendicular to one another.
42. Prove that the lines represented by $x^2 - 7xy + 12y^2 = 0$ are perpendicular to the lines represented by $12x^2 + 7xy + y^2 = 0$.
43. The equations to a pair of opposite sides of a rectangle are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0$. Find the equation of its diagonals.
44. Show that the four lines given by the equation $3x^2 + 8xy - 3y^2 = 0$ and $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$ form a square. Also find the equation of the diagonals of the square.
45. Prove that the equation $2x^2 - 5xy - 3y^2 - 2x + 6y = 0$ represents two straight lines and find their point of intersection.
46. Find the condition that one of the lines given by $ax^2 + 2hxy + by^2 = 0$ may be perpendicular to one of the lines given by $a_1x^2 + 2h_1xy + b_1y^2 = 0$.
47. If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ be $\lambda$ times the other, prove that $\left(\frac{1+\lambda}{2h}\right)^2 = \frac{\lambda}{ab}$.
48. If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ be the square of the other, prove that $\frac{a+b}{h} + \frac{8h^2}{ab} = 6$.
49. Find the condition that the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ and $a'x^2 + 2h'xy + b'y^2 = 0$ have one line in common.
50. Find the equation of the bisectors of angles between the lines $3x^2 - 5xy + 4y^2 = 0$.
51. Show that the straight lines represented by $135x^2 - 136xy + 33y^2 = 0$ are equally inclined to the line $x + 2y = 7$.
52. Prove that the lines $a^2x^2 + 2h(a + b)xy + b^2y^2 = 0$ are equally inclined to the lines $ax^2 + 2hxy + by^2 = 0$.
53. Show that the lines bisecting the angles between the bisectors of the angles made by lines $ax^2 + 2hxy + by^2$ are $(a - b)(x^2 - y^2) + 4hxy = 0$.
54. If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the other pair, prove that $pq = -1$.
55. Prove that the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 = \lambda(x^2 + y^2)$ are always the same irrespective of $\lambda$.
56. Prove that the bisectors of the angle between the lines $ax^2 + acxy + cy^2 = 0$ and $\left(3 + \frac{1}{c}\right)x^2 + xy + \left(3 + \frac{1}{a}\right)y^2 = 0$ are always the same.

57. Prove that the lines $2x^2 + 6xy + y^2 = 0$ are equally inclined to the lines $4x^2 + 18xy + y^2 = 0$.

58. If the lines represented by $x^2 - 2pxy - y^2 = 0$ are rotated about the origin through an angle $\theta$, one in clockwise direction and other in anti-clockwise direction, find the equation of the bisectors of the angles between the lines in the new position.

59. If one of the lines $ax^2 + 2hxy + by^2 = 0$ be the bisector of the angle between the coordinate axes, prove that $(a+b)^2 = 4h^2$.

60. Find the equation of pair of lines both of which pass through $(1, 2)$ and are parallel to the bisectors of the angles between the lines given by $x^2 + xy - 2y^2 + 4x - y + 3 = 0$.

61. Show that the lines joining the origin to the points common to $x^2 + hxy - y^2 + gx + fy = 0$ and $fx - gy = \lambda$ are at right angles irrespective of value of $\lambda$.

62. Prove that the angle between the lines joining the origin to the points of intersection of the straight line $y = 3x + 2$ with the curve $x^2 + 2xy + 3y^2 + 4x + 8y - 11 = 0$ is $\tan^{-1} \frac{2\sqrt{2}}{3}$.

63. Prove that the pair of lines joining the origin to the intersection of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by the line $lx + my + n = 0$ are conincident if $a^2l^2 + b^2m^2 = n^2$.

64. Show that the straight lines joining the origin to the other two points of intersection of the curves whose equations are $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x = 0$ will be at right angles if $g(a_1 + b_1) = g_1(a + b)$.

65. Find the equation of the straight lines joining the origin to the point of intersection of the line $3x + 4y = 5$ and the curve $2x^2 + 3y^2 = 5$.

66. Prove that the lines joining the origin and the points of intersection of the line $3x - 2y = 1$ and the curve $3x^2 + 5xy - 3y^2 + 2x + 3y = 0$ are perpendicular to each other.

67. Find the equation of the straight lines joining the origin to the points of intersection of the line $y = mx + c$ and the curve $x^2 + y^2 = a^2$. Prove that they are perpendicular to one another if $2c^2 = a^2(1 + m^2)$.

68. Find the equation of the straight lines joining the origin to the points of intersection of the line $lx + my + n = 0$ and $y^2 = 4ax$. Also find the condition of their perpendicularity.

69. Find the value of $c$ so that the lines joining the origin to the common points of $(x-3)^2 + (y-4)^2 = c^2$ and $4x + 3y = 24$ are at right angles.

70. Find the value of $m$, if the lines joining the origin and the points of intersection of $y = mx + 1$ and $x^2 + y^2 = 1$ are perpendicular to one another.

71. Prove that the straight lines joining the origin to the points of intersection of the straight lines $kx + hy = 2hk$ with the curve $(x-h)^2 + (y-k)^2 = c^2$ are at right angles if $h^2 + k^2 = c^2$.

72. Find the equation to the pair of lines through the origin and perpendicular to the pair of lines $ax^2 + 2hxy + by^2 = 0$.

73. Find the condition that the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ should be $\lambda$ times the slope of another.

74. Find the product of the length of the perpendiculars drawn from $(x_1, y_1)$ on the pair of straight lines $ax^2 + 2bxy + by^2 = 0$.

75. Prove that the equation $(a + 2b + b)x^2 - 2(a - b)xy + (a - 2h + b)y^2 = 0$ represents a pair of lines inclined at an angle of $45^{\circ}$ to one or other of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

76. If the distance of a given point $(\alpha, \beta)$ from each of the two straight lines through the origin is $d$, show that $(\alpha y - \beta x)^2 = d^2(x^2 + y^2)$.

77. If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ coincides with one of those given by $a_1x^2 + 2h_1xy + b_1y^2 = 0$ and the other lines represented by them be perpendicular, prove that $\frac{ba_1b_1}{b_1 - a_1} = \frac{hab}{b-a} = \frac{1}{2}\sqrt{-aa_1bb_1}$.

78. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represented a pair of parallel lines, prove that $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$. Also prove that the distance between these parallel lines is $2\sqrt{\frac{g^2 - ca}{a(a+b)}}$.

79. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, prove that the square of the distance of their point of intersection from the origin is $\frac{c(a+b) - f^2 - g^2}{ab - h^2}$.

80. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines equidistant from the origin, prove thta $f^4 - g^4 = c(bf^2 - ag^2)$.

81. If the lines $ax^2 + 2hxy + by^2 = 0$ be two sides of a parallelogram and the line $lx + my = 1$ be one of its diagonals, show that the equation of the other diagonal is $y(bl - hm) = x(am - hl)$.

82. Show that the orthocenter of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is given by $\frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2 - 2hlm + bl^2}$.

83. Find the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$.

84. Show that the straight lines $(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$ form with the line $Ax + By + C = 0$ an equilateral triangle whose area is $\frac{C^2}{\sqrt{3}(A^2 + B^2)}$.

85. Prove that the lines $(lx + my)^2 - 3(mx - ly)^2 = 0$ and $lx + my + n = 0$ form an equilateral triangle.

86. Show that the four straight lines given by $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ lie along the sides of a square.

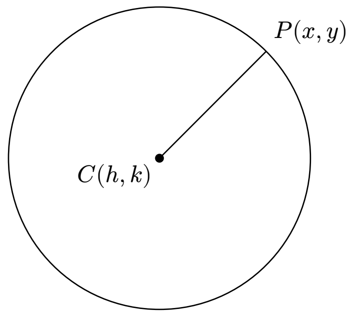
87. The lines represented by $x^2 - 3xy + 2y^2 = 0$ are shifted parallel to itself so that their point of intersection comes to $(1, 1)$. Find the combined equation of the lines in new position.

88. The joint equation of the lines of rays of incidence and reflection is $2x^2 - xy - y^2 = 0$. Find the joint equation of two possible lines from which the ray has been reflected.
89. If the angle between the lines joining the origin to the points of intersection of the lines $x \cos\alpha + y \sin\alpha = 1$ and the circle $x^2 + y^2 = a^2$ be $90^\circ$, find the possible values of $a$.
90. If the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ is rotated about the origin through $90^\circ$, find the equation in the new position.
91. Fing the image of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ by the line mirror whose equation is $y = 0$.
92. Prove that the sum of the squares of the perpendiculars drawn from the point $(x', y')$ on the lines given by $ax^2 + 2hxy + by^2 = 0$ is $\frac{[4h^2(x'^2+y'^2)+4h(a+b)x'y'+2(a-b)(ax'^2-by'^2)]}{[(a-b)^2+4h^2]}$.
93. If $(x, y)$ be the centroid of the triangle whose sides are the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$. Find the centroid.
94. A triangle has the line $ax^2 + 2hxy + by^2 = 0$ for two of its sides and the point $(l, m)$ for its orthocenter. Prove that the third side has the equation $(a + b)(lx + my) = am^2 - 2hlm + bl^2$.
95. Prove that the lines $x^2 + 4xy + y^2 = 0$ and $x - y = 4$ are the sides of an equilateral triangle. Find its area.
96. Find the internal angles of the triangle formed by the pair of straight lines $x^2 + 4xy + y^2 = 0$ and straight line $x + y + 4\sqrt{6} = 0$.
97. Prove that the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $x \cos\alpha + y \sin\alpha = p$ is $\frac{p^2\sqrt{h^2-ab}}{b\cos^2\alpha-2h\sin\alpha\cos\alpha+a\sin^2\alpha}$.
98. Find the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and the $x$-axis.

# 5 Circles

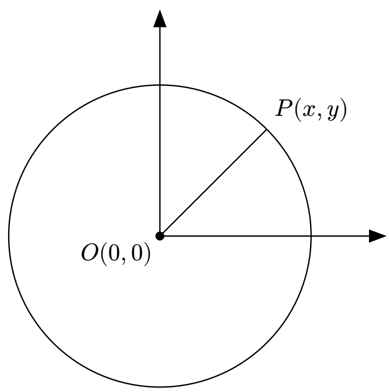
Circles are special cases of ellipses(we will study these the Chapter 7). For circles the length of major and minor axes of an ellipse are equal. Equivalently, a circle can be defined as locus of a point whose distance from a fixed center remains constant.

Consider a point $(h, k)$ as the center, the distance called radius as $a$ and the point $(x, y)$ then the general equation of the locus can be written as




$$(x-h)^2+(y-k)^2=a^2 \tag{5.1}$$

If we take the origin as the center then this equation reduces to




$$x^2+y^2=a^2 \tag{5.2}$$

In the last chapter we found the condition for general equation of second degree to represent a pair of straight lines. Here, we will find the condition for it to represent a circle.

Recall that general equaiton of second degree in $x$ and $y$ is given by $ax^2+2hxy+by^2+2gx+2fy+c=0$ and equation of circle is $(x-\alpha)^2+(y-\beta)^2=r^2$.

Comparing the coefficients we find that

$\frac{1}{a} = \frac{1}{b} = \frac{0}{h} = -\frac{\alpha}{g} = -\frac{\beta}{f} = \frac{\alpha^2+\beta^2-r^2}{c}$, hence coff. of $x^2$ should be equal to coefficient of $y^2$, coeff. of $xy$ should be zero.

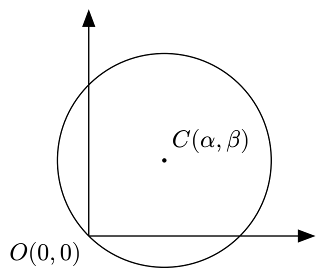
Thus, general equation in second degree which represents a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c \tag{5.3}$$

which will have center at $(-g, -f)$ and radius equal to $\sqrt{g^2 + f^2 - c}$.

## 5.1 Special Cases

**Case I:** The circle passes through the origin.




The equation of the circle is $(x-\alpha)^2 + (y-\beta)^2 = a^2$. Since it passes through the origin, therefore,

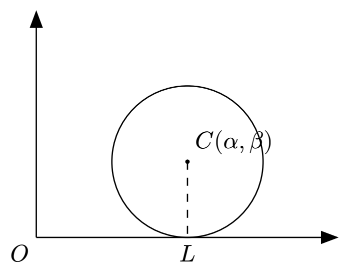
$(0-\alpha)^2 + (0-\beta)^2 = a^2$

$$\alpha^2 + \beta^2 = a^2 \tag{5.4}$$

If we consider general equation then we find that $c = 0$, thus the equation of circle is given by

$$x^2 + y^2 + 2gx + 2fy = 0 \tag{5.5}$$

**Case II:** The circle touches $x$-axis.



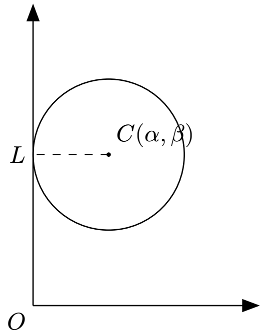

We see that $CL = |\beta|$ so we can write that

$$(x-\alpha^2) + (y-\beta)^2 = \beta^2 \tag{5.6}$$

In case we write it in form of general equation then $g^2 + f^2 - c = f^2$. Thus, the equation of circle becomes

$$x^2 + y^2 + 2gx + 2fy + g^2 = 0 \tag{5.7}$$

**Case III:** The circle touches the $y$-axis.




We see that $CL = |\beta|$ so we can write that

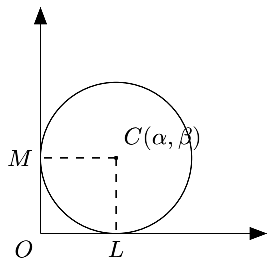
$$(x - \alpha^2) + (y - \beta)^2 = \beta^2 \tag{5.8}$$

In case we write it in form of general equation then $g^2 + f^2 - c = g^2$. Thus, the equation of circle becomes

$$x^2 + y^2 + 2gx + 2fy + f^2 = 0 \tag{5.9}$$

**Case III:** The circle touches both the axes.

We first consider the case of first quadrant.



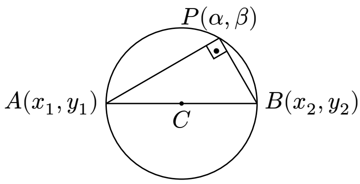

In this case $\alpha = \beta = a$, and hence, the equation becomes

$$(x - a)^2 + (y - a)^2 = a^2 \tag{5.10}$$

Similarly, in 2nd, 3rd and 4th quadrant the equation would be $(x + a)^2 + (y - a)^2 = a^2, (x + a)^2 + (y + a)^2 = a^2$ and $(x - a)^2 + (y + a)^2 = a^2$.

## 5.2 Circle on a Diameter

We will find equation of a circle one of whose diameters has endpoints as $(x_1, y_1)$ and $x_2, y_1$.




Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the endpoints of the diameter $AB$ of the circle as shown. Also, let $P(\alpha, \beta)$ on the circle. We know from geometry that $AB$ will subtend a right-angle on $P$.

Slope of $AP = \frac{\beta - y_1}{\alpha - x_1}$ and slope of $BP = \frac{\beta - y_2}{\alpha - x_2}$

Since these two lines are perpendicular to each other we have $\frac{\beta - y_1}{\alpha - x_1}\frac{\beta - y_2}{\alpha - x_2} = -1$

$\Rightarrow (\alpha - x_1)(\alpha - x_2) + (\beta - y_1)(\beta - y_2) = 0$

Thus, equation of circle would be

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \tag{5.11}$$

## 5.3 Parametric Form of a Circle

We have the equation of the circle as $(x - \alpha)^2 + (y - \beta)^2 = a^2$ then any point on the circle in parametric form can be given by $(\alpha + a\cos\theta, \beta + a\sin\theta)$.

If the center of the circle is at the origin i.e. the equation is $x^2 + y^2 = a^2$ then the point's coordiantes changes to $(a\cos\theta, b\sin\theta)$.

The point is often referred as point '$\theta$'.

## 5.4 Position of a Point w.r.t. a Circle

Consider any point $(x_1, y_1)$. For any circle the point will be inside, outside or on the circle as $(x_1 - \alpha)^2 + (y_1 - \beta)^2 > a^2, < a^2$ or $= a^2$.

For general second degree equation the condition would be $(x_1 + g^2) + (y_1 + f^2) > g^2 + f^2 - c, < g^2 + f^2 - c$ or $= g^2 + f^2 - c$.

## 5.5 Intersection of a Line and a Circle

We know that one of the equations of a straight line is $\frac{x - \alpha}{\cos\theta} = \frac{y - \beta}{\sin\theta}$. This equation represents a line passing through a point $P(\alpha, \beta)$ making an angle of $\theta$ with positive direction of $x$-axis. Let this ratio be equal to $r$, where $r$ is the algebraic distance of the point $(x, y)$ from $P(\alpha, \beta)$. So the coordiantes on this line are given by $(\alpha + r\cos\theta, \beta + r\sin\theta)$.

If this point lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, then

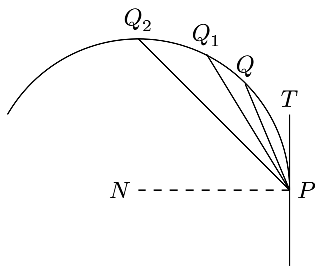
$(\alpha + r\cos\theta)^2 + (\beta + r\sin\theta)^2 + 2g(\alpha + r\cos\theta) + 2f(\beta + r\sin\theta) + c = 0$ i.e.

$$r^2 + 2r(\alpha\cos\theta + \beta\sin\theta + g\cos\theta + f\sin\theta) + \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c = 0 \quad (5.12)$$

This is a quadratic equaiton in $r$, and hence, the line through $P$ meets the circle at two points $A$ and $B$. Then we see that $PA.PB = \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c$, which is independent of $\theta$, i.e. the direction of the line.

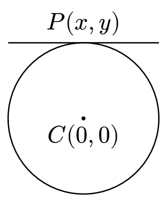
This we know from geometry that from a point $P$ a secant drawn to cut the circle in two points $A$ and $B$, the product of the distances $PA$ and $PB$ is constant.

## 5.6 Tangents and Normals




Consider two points $P$ and $Q$ on a curve. The position to which the line $PQ$ tends as $Q$ becomes closer to the point $Q$ i.e. the limiting position of chord $PQ$ as $Q$ tends to $P$ along the curve is called the *tangent* to the curve at point $P$. The point $P$ is called the point of contact of the tangent.

We will find equation of a tangent to a circle at a point $(x_1, y_1)$ on the circle $x^2 + y^2 = a^2$.




Let $Q(x_2, y_2)$ be another point on the circle. Then the equation of $PQ$ is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

Since bothe $P$ and $Q$ lie on the circle, therefore, $x_1^2 + y_1^2 = a^2$ and $x_2^2 + y_2^2 = a^2$

Subtracting we get $\frac{y_1 - y_2}{x_1 - x_2} = -\frac{x_1 + x_2}{y_1 + y_2}$

Substituting this in the equation of $PQ$ gives us

$y - y_1 = -\frac{x_1+x_2}{y_1+y_2}(x - x_1)$

As $Q \to P, x_2 \to x_1$ and $y_2 \to y_1$, and thus,

$y - y_1 = -\frac{x_1}{y_1}(x - x_1) \Rightarrow xx_1 + yy_1 = x_1^2 + y_1^2$. Thus,

$$xx_1 + yy_1 = a^2 \tag{5.13}$$

is required equation.

**Aliter:** We know that the tangent to a circle at a point in perpendicular to the radius through that point. The center of our circle is at the origin. Thus, slope of the radius is $\frac{y_1}{x_1}$.

So the slope of the tangent will be $-\frac{x_1}{y_1}$.

Thus, equation of tangent will be $y - y_1 = -\frac{x_1}{y_1}(x - x_1)$

$\Rightarrow xx_1 + yy_1 = a^2$.

**Aliter(Using Calculus):** The equation of our circle is $x^2 + y^2 = a^2$

Differentiating w.r.t. $x$ gives us $2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

Thus, slope at point $P(x_1, y_1)$ is $\frac{dy}{dx} = -\frac{x_1}{y_1}$

Thus, equation of tangent is $y - y_1 = -\frac{x_1}{y_1}(x - x_1)$

$\Rightarrow xx_1 + yy_1 = a^2$.

Now we will find tanegent to the circle represented by the equation $x^2 + y^2 + 2gx + 2fy + c = 0$.

Like the first method we see that both $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the circle. Thus, we can write

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ and $x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$

Subtracting $\frac{y_1-y_2}{x_1-x_2} = -\frac{x_1+x_2+2g}{y_1+y_2+2f}$

We know that equation of the line $PQ$ is $y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x - x_1)$

Substituting we get $y - y_1 = -\frac{x_1+x_2+2g}{y_1+y_2+2f}(x - x_1)$

Now as $Q \to P, x_2 \to x_1$ and $y_2 \to y_1$, which gives us

$y - y_1 = -\frac{x_1+g}{y_1+f}(x - x_1)$

$\Rightarrow xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$

$$xx_1 + yy_1 + g(x + x_1) + g(y + y_1) + c = 0 \tag{5.14}$$

**Aliter:** Center of the circle is $(-g, -f)$ and slope of radius is $\frac{y_1-(-f)}{x_1-(-g)} = \frac{y_1+f}{x_1+g}$

So slope of tangent would be $-\frac{x_1+g}{y_1+f}$

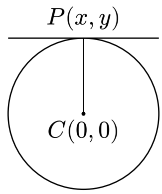
Thus, equation of tangent is $y - y_1 = -\frac{x_1+g}{y_1+f}(x - x_1)$

Proceeding like earlier the equation of tangent is found to be

$xx_1 + yy_1 + g(x + x_1) + g(y + y_1) + c = 0.$

You are encourage to try the third method using calculus for this as well.

Now we will find equation of the normal at $P(x_1, y_1)$ for the circle $x^2 + y^2 = a^2$. Normal is defined as perpendicular to tangent and it always passes through the center of the circle.




We have found the equation of tangent as $xx_1 + yy_1 = a^2$, and thus, its slope is $-\frac{x_1}{y_1}$, this makes the slope of the normal as $\frac{y_1}{x_1}$.

Since the normal pass through $C(0, 0)$, therefore, its equationis given by $y = \frac{y_1}{x_1}x$ i.e.

$$xy_1 - x_1y = 0 \tag{5.15}$$

You are implored to prove this by other techniques as well.

Similarly for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, the equation of the normal at $(x_1, y_1)$ is found to be

$$\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f} \tag{5.16}$$

## 5.7 Condition that a Line Touches a Circle

We will find the condition for a line $y = mx + c$ to touch the circle $x^2 + y^2 = a^2$.

Since the line touches the circle, therefore, $x^2 + (mx + c)^2 = a^2$

$\Rightarrow (1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0$

Since the line touches at one point the above equation will have a repeated root and the discriminant must be zero for that. Thus,

$4m^2c^2 - 4(1 + m^2)(c^2 - a^2) = 0 \Rightarrow c = \pm a\sqrt{1 + m^2}$. Thus the required condition is

$$c = \pm a\sqrt{1 + m^2} \tag{5.17}$$

Thus, general equation of tangent is $y = mx \pm a\sqrt{1 + m^2}$.

The equation of the line is $y = mx + c$ and equation of tangent is $xx_1 + yy_1 = a^2$

Since both equations represnet tangents at $(x_1, y_1)$, therefore, comparing the coefficients

$\frac{x_1}{-m} = \frac{y_1}{1} = \frac{a^2}{c} \Rightarrow x_1 = -\frac{a^2m}{c}, y_1 = \frac{a^2}{c}$

Thus, point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$.

### 5.7.1 Two Tangents to a Circle

We will prove that from a point outside the circle two tangents can always be drawn to the circle.

Let the circle be $x^2 + y^2 = a^2$. We know that the equation of tangent is $y = mx + a\sqrt{1+m^2}$

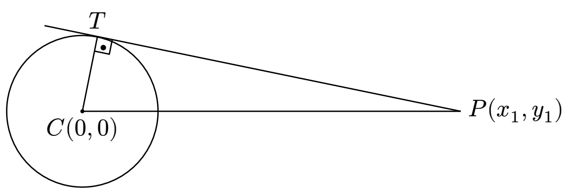
Let the external point be $(x_1, y_1)$. Since the tangent passes through it, therefore,

$y_1 = mx_1 + a\sqrt{1+m^2} \Rightarrow (y_1 - mx_1)^2 = a^2(1^2_m)$, which is a quadratic equation in $m$. Thus, it will give two values of $m$, and hence, two different tangents can be drawn from the same point.

If we check the nature of roots then discriminant is given by $x_1^2 + y_1^2 - a^2$ i.e. if the point is external then two real $m$s mean two different tangent.

### 5.7.2 Length of a Tangent

We will find the length of a tangent from an external point to a circle.




Let the point be $P(x_1, y_1)$ and the circle be $x^2 + y^2 = a^2$. From point $P$ draw a tangent to the circle, whihc touches the circle at $T$. $CT$ will be perpendicular to $PT$.

Thus, in right-angle $\triangle CPT$, we have

$PT^2 = CP^2 - CT^2 = x_1^2 + y_1^2 - a^2$. Thus,

$$PT = \sqrt{x_1^2 + y_1^2 - a^2} \tag{5.18}$$

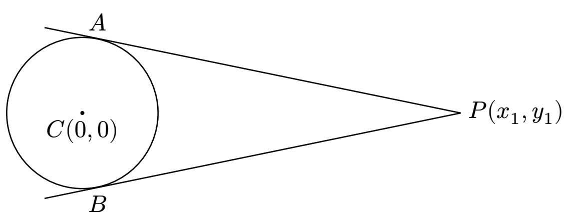
Similarly if the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the center is $(-g, -f)$ then

$PT^2 = (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$

$$PT = \sqrt{x_1^2 + y_2^2 + 2gx_1 + 2gy_1 + c} \tag{5.19}$$

### 5.7.3 Pair of Tangents

We will find the equation of pair of tangents drawn from an external point to a circle.



Let the point be $P(x_1, y_1)$ from which we draw two tangents $PA$ and $PB$ to the circle, touching the circle at $A$ and $B$ respectively.

Let $R(\alpha, \beta)$ be any point on any one of the tangents, say $PA$. Then the locus of $R(\alpha, \beta)$ will be the required equation of the pair of tangents to the circle from point $(x_1, y_1)$.

Equation of $PR$ is $y - y_1 = \frac{\beta - y_1}{\alpha - x_1}(x - x_1)$

$\Rightarrow y(\alpha - x_1) - x(\beta - y_1) - \alpha y_1 + \beta x_1 = 0.$

Now $AC = a$

$\Rightarrow \frac{|\beta x_1 - \alpha y_1|}{\sqrt{(\alpha - x_1)^2 + (\beta - y_1)^2}} = a \Rightarrow \frac{\beta x_1 - \alpha y_1}{(\alpha - x_1)^2 + (\beta - y_1)^2}) = a^2$

So locus of $R(\alpha, \beta)$ is $(yx_1 - xy_1)^2 = a^2\left[(x - x_1)^2 + (y - y_1)^2\right]$

$$(x_1^2 + y_1^2 - a^2)(x^2 + y^2 - a^2) = (xx_1 + yy_1 - a^2)^2 \tag{5.20}$$

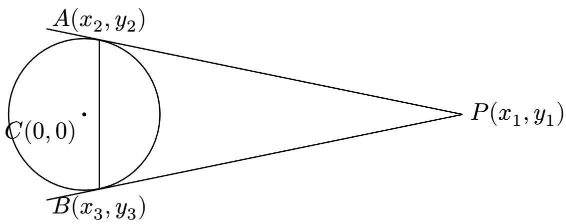
This is the required equation of pair of tangents drawn from $(x_1, y_1)$.

If circle is denoted by $S$ then the pair of tangents is given by $SS_1 = T^2$

For the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ the pair of tangents is given by

$$(x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c)(x^2 + y^2 + 2gx + 2fy + c) = [xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c]^2 \tag{5.21}$$

## 5.8 Chord of Contact of Tangents




Let $P$ be a point outside a circle. From $P$ two tangents $PA$ and $PB$ can be drawn to touch the circle at $A$ and $B$, respectively just like last section. The chord $AB$ is called the chord of contact for point $P$.

We will now find the equation of this chord of contact.

Let $A = (x_2, y_2)$ and $B = (x_3, y_3)$ as shown in the figure.

Since these points lie on the tangents, therefore, $xx_2 + yy_2 = a^2$ and $xx_3 + yy_3 = a^2$

Since both pass through $(x_1, y_1)$, therefore, $x_1x_2 + y_1y_2 = a^2$ and $x_1x_3 + y_1y_3 = a^2$

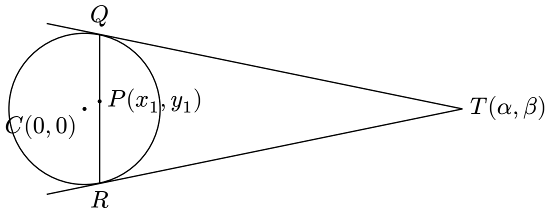
Thus, we can sat that $xx_1 + yy_1 = a^2$ passes through $A$ and $B$. Hence, the equation of line $AB$ is this equation.

Similarly, for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ the equation of chord of contact is given by

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

## 5.9 Poles and Polars

If from a point $P$ any straight line is drawn to meet the circle in $Q$ and $R$ and tangents at $Q$ and $R$ meet at $T$ then locus of $T$ is called the polar of $P$ w.r.t to the circle and $P$ is called the pole of its polar i.e. $T$.




The point $P$ can be inside or outside the circle. The diagram shows one such $P$ inside the circle through which the line $QR$ passes and tangents meets at $T(\alpha, \beta)$.

Clearly, $QR$ is the chord of contact of $T$ whose equation is $x\alpha + y\beta = a^2$, but it passes through $P(x_1, y_1)$, therefore,

$x_1\alpha + y_1\beta = a^2$

Putting $(x, y)$ instead of $(\alpha, \beta)$ we get the locus of $T$ as

$$xx_1 + yy_1 = a^2 \tag{5.22}$$

This is the required equation of the polar of point $P(x_1, y_1)$ and as can be seen from the equation it is a straight line.

Similarly, for circle $x^2 + y^2 + 2gx + 2fy + c = 0$ the equation of the polar is given by

$$xx_1 + yy_1 + g(x + x_1) + f(x + x_1) + c = 0 \tag{5.23}$$

### 5.9.1 Coordinates of a Pole

Now we will find coordinates of pole of a line.

Consider a line $lx + my + n = 0$ whose pole is to be found w.r.t. the circle $x^2 + y^2 = a^2$.

Let the pole be $(x_1, y_1)$. Now the equation of polar of point $(x_1, y_1)$ w.r.t. to the circle is given by $xx_1 + yy_1 = a^2$

Comapring coefficients with the given line $\frac{l}{x_1} = \frac{m}{y_1} = -\frac{n}{a^2} \Rightarrow x_1 = -\frac{a^2l}{n}, y_1 = -\frac{a^2m}{n}$

Hence, coordinates of the pole is given by $\left(-\frac{a^2l}{n}, -\frac{a^2m}{n}\right)$.

### 5.9.2 Properties of Poles and Polars

- If the polar of a point $P$ w.r.t. to a circle passes through $Q$ then the polar of $Q$ w.r.t. the circle will pass through $P$

  Let the equation of the circle is $x^2 + y^2 = a^2$. Also, let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$.

Now equations of the polars of these two points will be $xx_1 + yy_1 = a^2$ and $xx_2 + yy_2 = a^2$.

If the polar of $P$ passes through $Q$ then $x_1x_2 + y_1y_2 = a^2$.

Thus, we have proven our assumption. The points $P$ and $Q$ are called *conjugate points*.

- If the pole of a line w.r.t. a circle lies on another line, then the pole of the other line w.r.t. the same circle will lie on the first line.

  Let the circle be $x^2 + y^2 = a^2$ and the lines be $lx + my + n = 0$ and $l'x + m'y + n' = 0$.

  Let the pole of first line w.r.t to circle be $(x_1, y_1)$. Then the polar will be

  $xx_1 + yy_1 = a^2$. Comapring coefficients we arrive at $(x_1, y_1) = \left(-\frac{a^2l}{n}, -\frac{a^2m}{n}\right)$.

  This point will lie on the second line, therefore,

  $-\frac{a^2ll'}{n} - \frac{a^2mm'}{n} + n' = 0 \Rightarrow a^2ll' + a^2mm' - nn' = 0.$

  Similarly we let pole of the second line w.r.t. to the circle be $(x_2, y_2)$. Then the polar will be

  $xx_2 + yy_2 = a^2$. Comapring coefficients we arrive at $(x_2, y_2) = \left(-\frac{a^2l'}{n'}, -\frac{a^2m'}{n'}\right)$

  This point will lie on the first line, therefore,

  $a^2ll' + a^2mm' - nn' = 0.$

  Thus, the points lies on corresponding lines. Such lines are called *conjugate lines*.

- If the polars of two point $P$ and $Q$ w.r.t. a circle meet at $R$, then $R$ is the pole of the line $PQ$.

  Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points. Then the polar $AB$ and $CD$ of $P$ and $Q$ w.r.t. to the circle $x^2 + y^2 = a^2$ will be

  $xx_1 + yy_1 = a^2$ and $xx_2 + yy_2 = a^2$

  According to question they meet at $R$. Solving the two equations, we have

  $\frac{x}{y_1 - y_2} = \frac{y}{x_2 - x_1} = -\frac{a^2}{x_1y_2 - x_2y_1}$, which gives $R$.

  We have to prove that polar of $R$ w.r.t. the circle is line $PQ$. The polar of $R$ w.r.t. the circle is

  $x.\left(-\frac{a^2(y_1 - y_2)}{x_1y_2 - x_2y_1}\right) + y\left(-\frac{a^2(x_1 - x_1)}{x_1y_2 - x_2y_1}\right) = a^2$

  $\Rightarrow \frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$, which is the line $PQ$.

## 5.10 Equation of a Chord

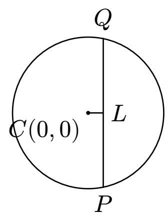
We will find equation of a chord whose midpoint is given.

Let the equation of the circle be $x^2 + y^2 = a^2$ with center $C(0, 0)$.

Let $PQ$ be the chord whose mid-point is $L(x_1, y_1)$. Slope of line $CL = -\frac{x_1}{y_1}$

Since $PQ \perp CL$, therefore, slope of $PQ = \frac{y_1}{x_1}$

Equation of $PQ$ is $y - y_1 = \frac{y_1}{x_1}(x - x_1)$



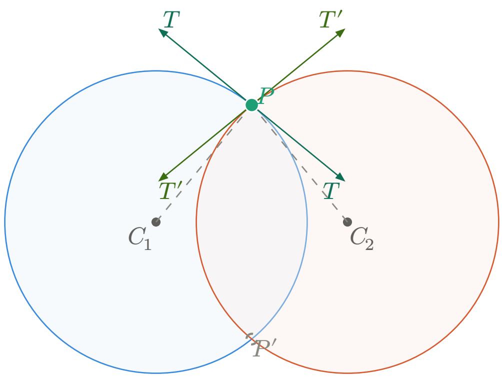

Thus, equation of the chord is given by

$$xx_1 + yy_1 = x_1^2 + y_1^2 \tag{5.24}$$

Similarly the equation of chord for the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \tag{5.25}$$

## 5.11 Intersection of Circles




The angle between two circles is the angle between their tangents at their point of intersection. Let $P$ be a point of intersection of two circles. Let $PT$ and $PT'$ be the tangents to the two circles at the point of intersection. Then the angle is defined at $\angle TPT'$ or $\pi - \angle TPT'$. If $C_1$ and $C_2$ be the centers of the two circles then

$PC_1 \perp PT$ and $PC_2 \perp PT'$ $\therefore \angle TPT' = \angle C_1PC_2$ or $\pi - \angle C_1PC_2$.

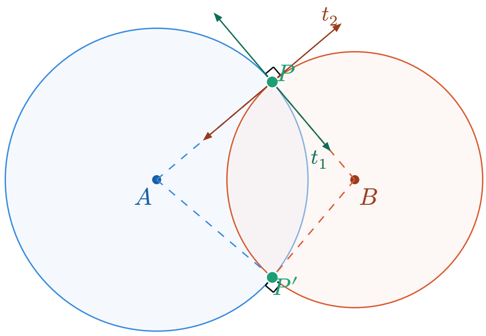
### 5.11.1 Orthogonal Circles

Two circles are said to intersect orthogonally if they intersect at right angles.

We will find the condition for two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ are orthogonal.

Let $A$ and $B$ be the center of these circles and $r_1$ and $r_2$ be their radii respectively. Then

$A = (-g_1, -f_1)$ and $B = (-g_2, -f_2)$ and $r_1^2 = g_1^2 + f_1^2 - c_1$ and $r_2^2 = g_2^2 + f_2^2 - c_2$



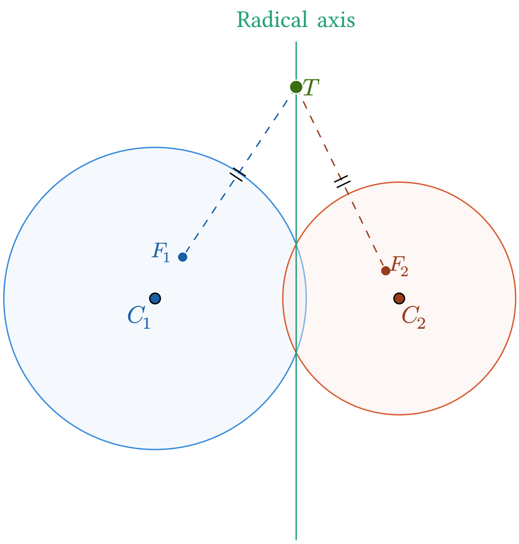

Orthogonality: $r_1^2 + r_2^2 = d^2$

Since the circles are orthogonal $PA^2 + PB^2 = AB^2 \Rightarrow r_1^2 + r_2^2 = AB^2$

$\Rightarrow g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2 = (g_2 - g_1)^2 + (f_2 - f_1)^2$

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2 \tag{5.26}$$

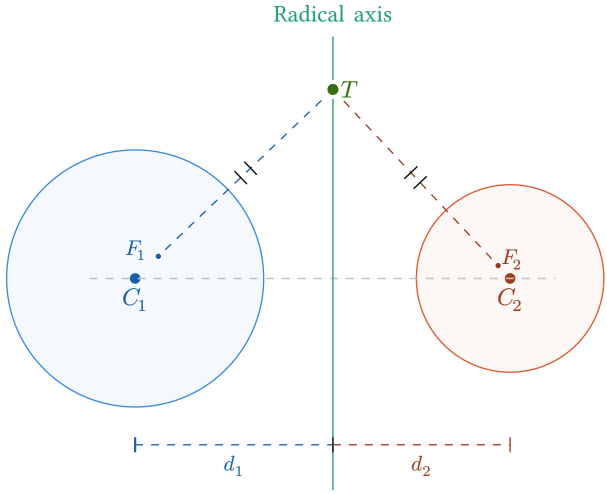
## 5.12 Radical Axis




The radical axis of two circles is the locus of a point which moves so that the length of the tangents drawn from it to the two points are equal.

We will find equation of radical axis of two circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.

Let $P(\alpha, \beta)$ be any point from which the tangents from it to these circles are equal. Let the tangents from this point to the two circles be $PA$ and $PB$.




Now $PA = \sqrt{\alpha^2 + \beta^2 + 2g_1\alpha + 2f_1\beta + c_1}$ and $PB = \sqrt{\alpha^2 + \beta^2 + 2g_2\alpha + 2f_2\beta + c_2}$

We have $PA = PB \Rightarrow 2(g_1 - g_2)\alpha + 2(f_1 - f_2)\beta + c_1 - c_2 = 0$

Hence, locus of $P$ is

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0 \tag{5.27}$$

which is a straight line and equation of radical axis.

### 5.12.1 Properties of Radical Axes

- The radical axis of two circles is perpendicular to the line of centers.

  Let the two circles be $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.

  Let $A$ and $B$ be the centers of these two circles. Then slope of $AB$ is $\frac{f_2 - f_1}{g_2 - g_1}$, which is negative reciprocal of $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$.

  Thus, the radical axis of two circles is perpendicular to the line of centers.
- The radical axis of three circles takes two at a time are concurrent.

  Consider three circles, which are $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$, $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and $x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$.

  So radical axes will be $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$, $2(g_2 - g_3)x + 2(f_2 - f_3)y + c_2 - c_3 = 0$ and $2(g_3 - g_1)x + 2(f_3 - f_1)y + c_3 - c_1 = 0$.

  Adding these we get equality, hence, the three lines are concurrent. The point where these radical axes meet is called the *radical center* of the three circles.
- The radical axis of the two circles bisect their common tangent.

  Let $AB$ be one of the common tangents meeting the raidcal axis at $P$, then since $P$ lies on the radical axis, hence, by definition of radical axis $PA = PB$.

  Thus, the radical axis bisect the common tangent.

- The locus of the cetner of a circle cutting two given circles orthogonally is the radical axis of the two circles.

  Let the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cut the circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ orthogonally. Then

  $2gg_1 + 2ff_1 = c + c_1$ and $2gg_2 + 2ff_2 = c + c_2$

  $\Rightarrow 2g(g_1 - g_2) + 2f(f_1 - f_2) = c_1 - c_2$

  Thus, locus of the center is $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$ which is the radical axis of the two circles being cut orthogonally.

## 5.13 Coaxial Circles

A system of circles is said to be coaxial if each pair of circles of the system has the same radical axis.

Consider tow circles $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.

Equation of radical axis will be $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$.

We know that the radical axis of two circles is perpendicular to the line joining their centers, therefore, if we take the line joining the centers as $x$-axis and radical axis as $y$-axis then

$f_1 = f_2 = 0$ and equation of radical axis will be $c_1 - c_2 = 0 \Rightarrow c_1 = c_2$.

Thus, constant term in all the circles must be same for them to be coaxial circles.

### 5.13.1 Limiting Points of a Coaxial System

Consider a system of coaxial circles having equation $x^2 + y^2 + 2gx + c = 0$, where $c$ is a constant and $g$ is a parameter.

The radius is $\sqrt{g^2 - c}$. If $g = \pm\sqrt{x}$ then the radius vanishes and the circle becomes point circle.

Thus, there are two circles of the system whose radii are zero, centers of these circles are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

These points are called *limiting points* of the system.

## 5.14 Circles through the Points of Intersection of a Circle and a Line

Consider a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and a line $lx + my + n = 0$.

Consider the equation $x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0$ which is also the equation of a circle.

The coordinates which satisfy the considered circle and line also satisfy this equation of circles. Thus, point of intersection of the considered circle and line lies on this new circle.

Similarly we can say that $x^2 + y^2 + 2gx + 2fy + c + \lambda(x^2 + y^2 + 2g_1x + 2f_1y + c) = 0$ is the locus of points of intersectionof two circles.

## 5.15 Problems

1. Find the center and the radius of the circle $3x^2 + 3y^2 - 8x - 10y + 3 = 0$.
2. Prove that the radii of the circles $x^2 + y^1 = 1, x^2 + y^2 - 2x - 6y = 6$ and $x^2 + y^2 - 4x - 12y = 9$ are in A.P.
3. Find the area of an equilateral triangle inscribed in the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.
4. Find the center and the radius of the circle $x^2 + y^2 + 6x - 8y = 24$.
5. Find the center and the radius of the circle $5x^2 + 5y^2 + 4x - 8y = 16$.
6. Find the center and the radius of the circle $x^2 + y^2 - 6x - 2y = 6$.
7. What will be the radius and center of the circle $\frac{1}{2}(x^2 + y^2) + x\cos\theta + y\sin\theta - 4 = 0$?
8. Prove that the centers of the circles $x^2 + y^2 = 1, x^2 + y^2 + 6x - 2y - 1 = 0$ and $x^2 + y^2 - 12x + 4y = 1$ are collinear.
9. Prove that the radii of the circles $x^2 + y^2 = 1, x^2 + y^2 - 2x - 6y = 6$ and $x^2 + y^2 - 4x - 12y = 9$ are in A.P.
10. Prove that the radii of the circles $x^2 + y^2 = 4, 4x^2 + 4y^2 - 8x - 24y + 15 = 0$ and $x^2 + y^2 - 4y - 5 = 0$ are in A.P.
11. Prove that the circles $x^2 + y^2 - 9 = 4r^2, r = 1, 2, 3$ cut off equal intercepts between the circles on the line $3x = 4y + 15 = 0$.
12. Find the equation of the circle whose center is $(1, 2)$ and which passes through the point $(4, 6)$.
13. If the equations of two diameters of a circle are $x + y = 6$ and $x + 2y = 4$ and the radius of the circle is 10, find the equation of the circle.
14. Find the equation of the circle whose center is $(3, 4)$ and which touches the line $5x + 12y = 1$.
15. Find the equation of a circle which passes through the point $(2, 0)$ and whose center is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ as $c$ tends to 1.
16. A circle has radius of 3 units and its center lies on the line $y = x - 1$. Find the equation of the circle if it passes through $(7, 3)$.
17. Find the equation of the circle, which touches the axes and whose center lies on the line $x - 2y = 3$.
18. A circle of radius 2 lies in the first quadrant and touches both the axes of coordinates. Find the equation of the circle with center at $(6, 5)$ and touching the above circle externally.

19. Find the equation of the circles whose center is $(3, -1)$ and which cut off an intercept of length 6 from $2x - 5y + 18 = 0$.
20. Find equations of the circles touching $y$-axis at $(0, 3)$ and making intercept of 8 units on the $x$-axis.
21. Find the equation of the circle having the pair of lines $x^2 + 2xy + 3x + 6y = 0$ as its normals and having the size just sufficient to contain the circle $x(x-4) + y(y-3) = 0$.
22. A circle of raidus 5 units touches the coordinate axes in the first quadrant. If the circle makes one complete roll on the $x$-axis along the positive direction of $x$-axis, find its equation in the new position.
23. The circle $x^2 + y^2 - 4x - 8y + 16 = 0$ rolls up the tangent to it at $\left(2 + \sqrt{3}, 3\right)$ by 2 units, assuming the $x$-axis as horizontal, find the equation of the circle in the new position.
24. Find the equations of the circles touching the lines $y = 0$ and $y = \sqrt{3}(x+1)$ and having the centers at a distance 1 from the origin.
25. Find the equation of the circle passing through the point $(2, 8)$ touching the lines $4x - 3y + 24 = 0$ and $4x + 3y - 42 = 0$ and having $x$ coordinate of the center of the circle numerically less than or equal to 8.
26. Find the equation of the circle whose center is $(1, -5)$ and radius is 7.
27. Find the equation of the circle whose center is $(-1, -2)$ nad diameter is 25.
28. If the equation of two diameters of a circle are $2x + y = 6$ and $3x + 2y = 4$ and the radius is 10, find the equation of the circle.
29. Find the equation of the circle which passes through the point of intersection of $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$ and whose center is $(2, 3)$.
30. Find the equation of the circle whose center is $(1, 2)$ and which passes through the point of intersection of $3x + y = 14$ and $2x + 5y = 18$.
31. Find the equation of the circle passing through the center of the circle $x^2 + y^2 - 4x - 6y = 8$ and being concentric with the circle $x^2 + y^2 - 2x - 8y = 5$.
32. Find the equation of the circle passing through the point of intersection of $x + 3y = 0$ and $2x - 7y = 0$ and whose center is the point of intersection of lines $x + y + 1 = 0$ and $x - 2y + 4 = 0$.
33. Find the equation of the circle whose radius is 5 and the center lies on the positive side of $x$-axis at a distance 5 from the origin.
34. Find the equation of the circle which passes through the points $(-1, 2)$ and $(3, -2)$ and whose center lies on the line $x - 2y = 0$.
35. Find the equation of the circle which passes through the point $(1, -2)$ and $(4, -3)$ and whose center lies on the line $3x + 4y = 7$.
36. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point $(5, 5)$.

37. Find the equation of the circle whose center is $(1, -3)$ and which touches the line $2x - y - 4 = 0$.
38. Write down the equation of a circle concentric with the circle $x^2 + y^2 - 4x + 6y - 17 = 0$ and tangent to the line $3x - 4y + 7 = 0$.
39. Find the equation of the circle of radius 5 and touching the line $3x - 4y + 5 = 0$ at $(1, 2)$.
40. If the raidus of the circle is 5 and the equations of the twp normals to the circle are $3x - 5y + 2 = 0$ and $x + 2y = 3$, find the equation of the circle.
41. Find the equation of the circle which touches the $y$-axis at a distance 4 from the origin towards the positive side of $y$-axis and cuts of an intercept 6 on the $x$-axis.
42. Find the equation of the circle which touches both the axes and whose radius is $a$.
43. Find the equation of the circle passing through the point $(h, k)$ and touching the $y$-axis at origin.
44. Find the equation of the circle touching the axis of $x$ at the origin and touching the line $3y = 4x + 24$.
45. What is the parametric equation of the circle $x^2 + y^2 = 16$.
46. Find the equation of the circle which touches the line $2x - y = 1$ at $(1, 1)$ and the line $2x + y = 4$.
47. Find the euqation of the circle touching the lines $4x - 3y = 30$ and $4x - 3y + 10 = 0$ having the center on the line $2x + y = 0$.
48. A circle of raidus 4 units touches the coordinate axes in the first quadrant . Find the equation of its image w.r.t. the line mirror $y = 0$.
49. The equation of a circle is $x^2 + y^2 + 16x - 24y + 183 = 0$. Find the equation of the image of this circle by the line mirror $4x + 7y + 13 = 0$.
50. The circle $(x - a)^2 + (y - a)^2 = a^2$ is rolled on the $x$-axis in the positive direction through one complete revolution. Find the equation of the circle in the new position.
51. The center of a circle is $(1, 1)$ and its radius is 5 units. If the center is shifted on the line $x - y = 0$ through a distance of $\sqrt{2}$ units, find the equation of the circle in the new position. How many such circles are possible?
52. Find the equation of the circles which pass through the origin and cut off equal chords of $\sqrt{2}$ units from the straight lines $y = x$ and $y = -x$.
53. Find the center of the circle which is inscribed in the triangle formed by the lines $3x + 4y - 15 = 0, 3x - 4y - 7 = 0$, and $12x + 5y - 115 = 0$.
54. Show that the equation of the circle which touches the coordinate axes and whose center lies on the straight line $lx + my + n = 0$ is $(l \pm m)^2(x^2 + y^2) \pm 2n(l \pm m)(x + y) + n^2 = 0$.

55. Of the two concentric circles the smaller one has the equation $x^2 + y^2 = 4$. If the intercept on the line $x + y = 2$ made between the two circles is 1, find the equation of the larger circle.
56. Is $(3, 2)$ an interior point or an exterior point of the circle $x^2 + y^2 - 2x + y = 0$? If inerior, find the equation of the circle centered on it and of maximum area contained in the given in the circle. If exterior, find the circle of maximum radius centered on it containing the given circle.
57. Find the equation of the circle with minimum raidus which contains the three circles $x^2 + y^2 - 4y - 5 = 0, x^2 + y^2 + 12x + 4y + 31 = 0,$ and $x^2 + y^2 + 6x + 12y + 36 = 0$.
58. Find the equation of the circle whose diameter is the line joining the points $(-4, 3)$ and $(12, -1)$. Find also the intercept made by it on $y$-axis.
59. The sides of a square are $x = 1, x = 3, y = 2$ and $y = 4$. Find the equation of the circle drawn on the diagonals of the square as its diameter.
60. Find the equation of the circle on the line joining the origin and $(2, -4)$ as diameter.
61. Find the equation of the circle, the endpoints of whose diameter are $(2, -3)$ and $(-2, 4)$. Find its center and radius.
62. Find the equation of the circle drawn on the intercept between the axes made by the line $3x + 4y = 12$ as a diameter.
63. Find the equation of the circle the endpoints of whose diameter are the centers of the circles $x^2 + y^2 + 6x - 14y = 1$ and $x^2 + y^2 - 4x + 1 - y = 2$.
64. The sides of a square are $x = 6, x = 9, y = 3$ and $y = 6$. Find the equation of a circle drawn on the diagonal of the square as its diameter.
65. Find the equation of the circle circumscribing the rectangle whose sides are $x - 3y = 4, 3x + y = 22, x - 3y = 14$ and $3x + y = 62$.
66. The ascissa if two points $A$ and $B$ are the roots of the equation $x^2 + 2x - a^2 = 0$ and the oridinates are the roots of the equation $y^2 + 4y - b^2 = 0$. Find the equation of the circle with $AB$ as its diameter. Also find the coordinates of the center and the length of the raidus of the circle.
67. If $(4, 1)$ be one endpoint of the diameter of the circle $x^2 + y^2 - 2x + 6y - 15 = 0$, find the coordinates of the other endpoint of the diagonal.
68. Find the equation of the circumcircle of the quadrilateral formed by the four lines $ax + by \pm c = 0$ and $bx - ay \pm c = 0$.
69. Find the equation of the circle which passes through the points $(1, -2)$ and $(4, -3)$ and whose center lies on the line $3x + 4y = 7$.
70. Find the equation of the circle passing through the origin and the points where the straight line $3x + 4y = 12$ meets the coordinate axes.
71. Show that the cyclic quadrilateral is formed by the lines $5x + 3y = 9, x = 3y, 2x = y$ and $x + 4y + 2 = 0$ taken in order. Find the equation of the circle.

72. Find the equation of a circle passing through the points $(1, 2)$ and $(3, 4)$ and touching the line $3x + y - 3 = 0$.

73. A circle touches both the $x$-axis and the line $4x - 3y + 4 = 0$. Its center is in the third quadrant anf lies on the line $x - y - 1 = 0$. Find the equation of the circle.

74. Find the equation of the circle passing through the points $(1, 0), (0, 1)$ and $(1, -2)$.

75. Find the equation of the circle passing through the points $(0, 0), (a, 0)$ and $(0, b)$.

76. Find the equation of the circle which passes through the origin and cuts off chords of length 4 and 6 on the positive side of $x$-axis and $y$-axis respectively.

77. Find the equation of the circumcircle of the triangle formed by the lines $y = x, y = 2x$, and $y = 3x + 2$.

78. Find the incenter of the triangle whose sides are $7x - y + 11 = 0, x + y - 15 = 0$ and $7x + 17y + 65 = 0$. Find the equation of the incircle.

79. Find the equation of the circles passing through the origin and cutting off equal intercepts 1 from the lines $3x = 4y$ and $4x = 3y$.

80. Find the equation of the circle described on the common chord of the circles $x^2 + y^2 - 4x - 5 = 0$ and $x^2 + y^2 + 8y + 7 = 0$ as diameter.

81. Show that the circle on the chord $x \cos\alpha + y \sin\alpha - p = 0$ of the circle $x^2 + y^2 = a^2$ as diameter is $x^2 + y^2 - a^2 - 2p(x \cos\alpha + y \sin\alpha - p) = 0$.

82. Prove that the equation $x^2 + y^2 - 4 + k\left(y - mx - 2\sqrt{1 + m^2}\right) = 0$ represents a family of circles touching each other at a common point for all $k$, where $m$ is a given constant.

83. Show that the general equation of a circle which passes through the points $(x_1, y_1)$ and $(x_2, y_2)$ may be written as $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$, and hence, deduce the diameter form of the equation of a circle.

84. The line $Ax + By + C = 0$ cuts the circle $x^2 + y^2 + ax + by + c = 0$ in $P$ and $Q$. The line $A'x + B'y + C' = 0$ cuts the circle $x^2 + y^2 + a'x + b'y + c' = 0$ in $R$ and $S$. If $P, Q, R, S$ are concyclic points show that $\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$.

85. A fixed circle is cut by circles passing through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$. Show that the chord of intersection of the fixed circle with any one of the circled, passes through a fixed point.

86. Tangents $PQ$ and $PR$ are drawn to the circle $x^2 + y^2 = a^2$ from the point $P(x_1, y_1)$. Find the equation of the circumcircle of $\triangle PQR$.

87. Find the equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$ and $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its center on the line $y = x$.

88. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$, prove that $2g'(g - g') + 2f'(f - f') = c - c'$.

89. Find the equation of the circle of radius 4 and passing through the point of intersection of circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$.

90. Show that the common chord of the circles $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 8x - 6y + 23 = 0$ pass through the center of the second circle and find its length.

91. Find the equation of the circle whose diameter is the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.

92. If $y = mx$ be the equation of a chord of the circle $x^2 + y^2 - 2ax = 0$, prove that the circle of which this chord is a diameter has the equation $(1 + m^2)(x^2 + y^2) - 2a(x + my) = 0$.

93. Find the equation of the circle which passes through the point of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$ and $x^2 + y^2 + 2x - 4y - 6 = 0$ and whose center lies on the line $y = x$.

94. Prove that the equation $x^2 + y^2 + 2(3 + p)x + 2(3 - p)y + 4 = 0$ represents a circle for all values of $p$, passing through two fixed points. Find the fixed points.

95. Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 = 4a^2$ and $x^2 + y^2 - 2x - 4y + 4 = 0$ and touching the line $x + 2y = 0$.

96. Find the equation of the circle which passes through the points of intersection of the circle $x^2 + y^2 - x - y = 0$ and the line $x + y = 1$ and also through the point $(1, 1)$.

97. Find the equation of the circle which has for its diameter the chord cut off on the line $px + qy - 1 = 0$ by the circle $x^2 + y^2 = a^2$.

98. The point $A(\alpha, \beta)$ is outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and $AP, AQ$ are tangents to the circle. Find the equation of the circumcircle of $\triangle\, PQR$.

99. Show that the line $3x - 4y = c$ will meet the circle having center at $(2, 4)$ and the radius 5 in real and distinct points if $-35 < c < 15$.

100. Find the length of the chord $4x - 3y = 5$ of the circle $x^2 + y^2 + 3x - y - 10 = 0$.

101. Prove that the line $y = x + a\sqrt{2}$ touches the circle $x^2 + y^2 = a^2$. Also find the point of contact.

102. Find the equation of tangents of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ which are parallel to the line $4x + 3y + 5 = 0$.

103. Show that the common tangents to the circles $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 + 2x = 0$ form an equilateral triangle.

104. Three concentric circles of which the biggest is $x^2 + y^2 = 1$ have their radii in A.P. If the line $y = x + 1$ cuts all the circles in real and distinct points then find the interval in which the common difference of the A.P. will lie.

105. If $4l^2 - 5m^2 + 6l + 1 = 0$; then shown that the line $lx + my + 1 = 0$ touches a fixed circle. Find the center and raidus of the circle.

106. Find the point $P$ on the circle $x^2 + y^2 - 4x - 6y + 9 = 0$ such that (i) $\angle POX$ is minimum (ii) $OP$ is maximum, where $O$ is the origin and $OX$ is the $x$-axis.

107. Show that the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touches both the axes. Also, find the point of contact.

108. Find the length of the chord of the circle $x^2 + y^2 - 16$ which bisects the join of the points $(2, 3)$ and $(1, 2)$ perpendicularly.

109. Find the length of the chord intercepted by the straight line $x - 7y + 4 = 0$ and the circle $x^2 + y^2 - 14x + 4y + 28 = 0$. Also, find the middle point of the chord.

110. Find the length of the common chords of the circles $x^2 + y^2 + 3x + 5y + 4 = 0$ and $x^2 + y^2 + 5x + 3y + 4 = 0$.

111. Find the equation and length of the common chord of the circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.

112. Prove that the length of the common chord of the circlea $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$ is $\sqrt{4c^2 - 2(a - b)^2}$. Hence find the condition that the two circles may touch each other.

113. Prove that the length of the common chord of the two circles $x^2 + y^2 + 2hx + a^2 = 0$ and $x^2 + y^2 - 2ky - a^2 = 0$ is $2\sqrt{\frac{(h^2-a^2)(k^2+a^2)}{h^2+k^2}}$.

114. Prove that the length of the common chord of the circles $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$ is $\sqrt{\frac{1}{2}(a + b)^2 - 4c}$.

115. If the line $px + qy + r = 0$ touches the circle $x^2 + y^2 = a^2$ then prove that $r^2 = a^2(p^2 + q^2)$.

116. Prove that the line $4x - 3y + 23 = 0$ touches the circle $x^2 + y^2 - 6x + 10y = 66$.

117. Show that for all values of $\theta$, $x \sin\theta - y \cos\theta = a$ touches the circle $x^2 + y^2 = a^2$.

118. If $lx + my = 1$ touches the circle $x^2 + y^2 = a^2$, prove that the point $(l, m)$ lies on the circle $x^2 + y^2 = a^{-2}$.

119. Find the value of $\lambda$ so that the line $3x - 4y = \lambda$ may touch the circle $x^2 + y^2 - 4x - 8y - 5 = 0$.

120. Show that the line $(x - 1)\cos\theta + (y - 1)\sin\theta = 1$ touches a circle for all values of $\theta$. Find the circle.

121. Find those tangents to the circle $x^2 + y^2 = 16$ which are parallel to $3x - 16y = 10$.

122. Find the equation of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are (i) parallel, (ii) perpendicular to the line $3x - 4y - 1 = 0$.

123. Show that the line $7y - x = 5$ touches the circle $x^2 + y^2 - 5x + 5y = 0$ and find the equation of the other parallel tangent.

124. Find the equation of the tangent lines to the circle $x^2 + y^2 = 15$ which are perpendicular to the line $4x - y + 6 = 0$.

125. Find the equation of tangents to the circle $x^2 + y^2 - 6x + 4y - 3 = 0$, which are perpendicular to the line $y = 2x - 1$.

126. Find the equation of the tangents to the circle $x^2 + y^2 = 25$, which make an angle of $60^\circ$ with the positive direction of $x$-axis.
127. Find the equation of the family of circles which touch the pair of straight lines $x^2 - y^2 + 2y - 1 = 0$.
128. Examine if the two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$ touch each other externally or internally.
129. Prove that the circle $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other if $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.
130. Prove that $x^2 + y^2 = a^2$ and $(x - 2a)^2 + y^2 = a^2$ are two equal circles touching each other. Find the equation of circles of equal radius touching both the circles.
131. Prove that the circles $x^2 + y^2 + 2x + 2y + 1 = 0$ and $x^2 + y^2 - 4x - 6y - 3 = 0$ touch each other.
132. Prove that the condition for the circles $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$ to touch each other is $a = b \pm \sqrt{2}c$.
133. Prove that the circles $x^2 + y^2 + 2ux + 2vy = 0$ and $x^2 + y^2 + 2u_1x + 2v_1y = 0$ touch each other if $uv_1 = u_1v$.
134. Find the length of the chord of the circle $x^2 + y^2 = 4$ through $\left(1, \frac{1}{2}\right)$, which is of minimum length.
135. Find the angle that the chord of circle $x^2 + y^2 - 4y = 0$ along the line $x + y = 1$ subtends at the circumference the larger segment.
136. Prove, analytically, that the angle n a semi-circle is a right angle.
137. A circle of diameter 13m with the center $O$ coinciding with the origin of coordinate axes, has the diameter $AB$ on the $x$-axis. If the length of the chord $AC$ be 5m, find the equation of pair of lines $BC$, $C$ having two possible positions.
138. Show that the least chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which passes through the internal point $(\alpha, \beta)$ is equal to $2\sqrt{-(\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c)}$.
139. If the line $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points prove that $a_1a_2 = b_1b_2$.
140. The chord along the line $y - x = 3$ of the circle $x^2 + y^2 = k^2$, subtends an angle of $30^\circ$ in the major segment of the circle cut off by the chord. Find $k$.
141. Prove that the tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ and find its point of contact.
142. Prove that the equation $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$, where $\lambda$ is a parameter, represents a family of circles passing through two fixed points $A$ and $B$ on the $x$ -axis. Also, find the equation of that circle of the family the tangents to which at $A$ and $B$ meet on the line $x + 2y + 5 = 0$.
143. $AB$ is a diameter of a circle. $CD$ is a chord parallel to $AB$ and $2CD = AB$. The tangent at $B$ meets the line $AC$ produced at $E$. Prove that $AE = 2AB$.
144. Two parallel tangents to a given circle are cut by a third tangent at point $A$ and $B$. If $C$ be the center of the given circle, prove that $\angle ACB$ is a right angle.

145. A circle of radius 5 meters is having its center at $A$ at the origin. Two circles II and III with centers $B$ and $C$ of radii 3 and 4 meters respectively touch the circle I and also touch the $x$-axis to the right of $A$. Find the equation to any two common tangents to the circles II and III.

146. Find the equation of the normal to the circle $x^2 + y^2 - 2x - 4y + 3 = 0$ at the point $(2, 3)$.

147. The extremeties of a diagonal of a rectangle are $(-4, 4)$ and $(6, -1)$. A circle circumscribes the rectangle and cuts an intercept $AB$ on the $y$-axis. Find the area of the triangle formed by $AB$ and the tangents to the circle at $A$ and $B$.

148. Find the equation of the tangent to the circle $x^2 + y^2 - 4x - 6y = 12$ at the point $(-1, -1)$.

149. Prove that the tangents to the circle $x^2 + y^2 - 7x - 5y + 18 = 0$ at the point $(4, 3)$ and $(3, 2)$ are parallel to each other.

150. Find the equations of the tangents to the circle $x^2 + y^2 = 169$ at $(5, 12)$ and $(12, -5)$ and prove that they cut at right angles. Find their point of intersection also.

151. The tangent at the point $(\alpha, \beta)$ to the circle $x^2 + y^2 = r^2$ cut the axes of coordinates in $A$ and $B$. Prove that the area of the $\triangle\, OAB$ is $\frac{1}{2}\frac{r^4}{|\alpha\beta|}$, $O$ being the origin.

152. Let $A$ be the center of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the point $B(1, 7)$ and $D(4, -2)$ on the circle meet at the point $C$. Find the area of the quadrilateral $ABCD$.

153. Prove that the line $x + y = 5$ touches the circle $x^2 + y^2 - 2x - 4y + 3 = 0$. Find the point of contact.

154. Prove that the tangent to the circle $x^2 + y^2 = 5$ at the point $(1, -2)$ also touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$, and find its point of contact.

155. Show that the circles $x^2 + y^2 - 10x + 4y - 20 = 0$ and $x^2 + y^2 + 14x - 6y + 22 = 0$ touch each other. Find the coordinates of the point of contact and the equation of the common tangent at the point of contact.

156. Prove that the line $y = x + 2$ touches the circle $x^2 + y^2 = 2$. Find the point of contact.

157. Find the condition that the straight line $lx + my + n = 0$ should touch the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and also find the coordinates of the point of contact.

158. If the line $3x + 4y = k$ touches the circle $x^2 + y^2 = 10x$, find the value of $k$ and also the point of contact.

159. Find the equation of the normal to the circle $x^2 + y^2 = 5$ at the point $(1, 2)$.

160. Find the equation of the normal to the circle $x^2 + y^2 = 2x$, which is parallel to the line $x + 2y = 3$.

161. The point $(1, 4)$ is inside the circle $S$ whose equation is of the form $x^2 + y^2 - 6x - 10y + k = 0$. What are the possible values of $k$ if the circle $S$ neither touches the axes not cuts the them?

162. Find the length of the tangent drawn from the point $(5,1)$ to the circle $x^2+y^2+6x-4y-3=0$.

163. Distances from the origin to the centers of the three circles $x^2+y^2-2\lambda x=c^2$, where $c$ is a constant and $\lambda$ a vairable are in G.P. Prove that the lengths of tangents drawn from any point on the circle $x^2+y^2=c^2$ to the three circles are also in G.P.

164. From a point $P$ tangents drawn to the circles $x^2+y^2+x-3=0, 3x^2+3y^2-5x+3y=0$ and $4x^2+4y^2+8x+7y+9=0$ are of equal lengths. Find the equation of the circle thorugh $P$ which touches the line $x+y=5$ at the point $(6,-1)$.

165. If the length of the tangent from $(f,g)$ to the circle $x^2+y^2=6$ be twice the length of the tangent from $(f,g)$ to the circle $x^2+y^2+3x+3y=0$ then will $f^2+g^2+4f+4g+2=0$?

166. If the length of the tangent from a point $(f,g)$ to the circle $x^2+y^2=4$ be four times the length of the tangent from it to the circle $x^2+y^2=4x$, show that $15f^2+15g^2-64f+4=0$.

167. Show that the length of the tangent from any point on the circle $x^2+y^2+2gx+2fy+c=0$ to the circle $x^2+y^2+2gx+2fy+c_1=0$ is $\sqrt{c_1-1}$.

168. Find the point from which the tangents to the three circles $x^2+y^2=1, x^2+y^2-8x+15=0$ and $x^2+y^2+10y+24=0$ are equal in length.

169. Find the point from which the tangents to the three circles $x^2+y^2-4x+7=0, 2x^2+2y^2-3x+5y+9=0$ and $x^2+y^2+y=0$ are equal in length. Find the length as well.

170. If $A_i$ is the center of the circle $x^2+y^2+2g_ix+5=0$ and $t_i$ is the length of the tangents from any point to this circle $i=1,2,3$; then show that $(g_2-g_3)t_1^2+(g_3-g_1)t_2^2+(g_1-g_2)t_3^2=0$.

171. Show that if the length of the tangent from a point $P$ to the circle $x^2+y^2=a^2$ be four times the length of the tangent from it to the circle $(x-a)^2+y^2=a^2$, then $P$ lies on the circle $15x^2+15y^2-32ax+a^2=0$.

172. Find the equation of the pair of tangents drawn to the circle $x^2+y^2-2x+4y=0$ from point $(0,1)$.

173. If from any point on the circle $x^2+y^2+2gx+2fy+c=0$, tangents are drawn to the circle $x^2+y^2+2gx+2fy+c\sin^2\alpha+\left(g^2+f^2\right)\cos^2\alpha=0$, show that the angle between the tangents is equal to $2\alpha$.

174. Find the equations of tangents to the circle $x^2+y^2=25$ which pass through $(1,-7)$ and show that they are at right angles.

175. Show that two tangents can be drawn from the point $(9,0)$ to the circle $x^2+y^2=16$; also find the equation of the pair of tangents and the angle between them.

176. Find the equations of the tangents through $(7,1)$ to the circle $x^2+y^2=25$.

177. Find the equation of the pair of tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + k^2 = 0$, and show that their intercept on the line $y = h$ is $\frac{2hk}{k^2-g^2}$ times the radius of the circle.

178. Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y - 11 = 0$, whose middle point is $(1, -1)$.

179. Find the equation of the chord of the circle $x^2 + y^2 + 6x + 8y + 9 = 0$ whose mid-point is $(-2, -3)$.

180. Find the coordinates of the mid-point of the chord which the circle $x^2 + y^2 + 4x - 2y - 3 = 0$ cuts off on the line $y = x + 2$.

181. Find the equation of the chord fo contact of the tangents drawn from $(1, 2)$ to the circle $x^2 + y^2 - 2x + 4y + 7 = 0$.

182. Tangents are drawn from the point $(h, k)$ to the circle $x^2 + y^2 = a^2$. Prove that the area of the triangle formed by them and their chord of contact is $\frac{a(h^2+k^2-a^2)^{\frac{3}{2}}}{h^2+k^2}$.

183. The chord of contact of tangents from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $c^2 + y^2 = c^2$. Show that $a, b, c$ are in G.P.

184. Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$; find the points of intersection of these tangents.

185. Find the equation of the chord of contact of the tangents drawn from an external point $(-3, 2)$ to the circle $x^2 + y^2 + 2x - 3 = 0$.

186. Find the equation of the chord of contact of the tangents drawn from $(5, 3)$ to the circle $x^2 + y^2 = 25$.

187. Find the coordinates of the point of intersection of tangents at the points where the line $2x + y + 12 = 0$ meets the circle $x^2 + y^2 - 4x + 3y - 1 = 0$.

188. The lnegth of tangents from the two given points to a given points to a given circle are $t_1$ and $t_2$. If the tow given points are conjugate to each other w.r.t. the given circle, prove that the distance between the points will be $\sqrt{t_1^2 + t_2^2}$.

189. Find the area of the triangle formed by the tangents drawn from the point $(4, 6)$ to the circle $x^2 + y^2 = 25$ and their chord of contact.

190. If $OP$ and $OQ$ are the tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, where $P$ and $Q$ are the points of contact, show that the equation of the circumcircle of the triangle $OPQ$ is $x^2 + y^2 + gx + fy = 0$.

191. Find the point of contact of the tangents to the circle $x^2 + y^2 = 25$ that pass through the point $(7, 1)$ and give the equation of tangents.

192. Find the equation of the polar of the point $(2, -1)$ w.r.t. the circle $x^2 + y^2 - 3x + 4y - 8 = 0$.

193. Find the pole of the line $3x + 5y + 17 = 0$ w.r.t. the circle $x^2 + y^2 + 4x + 6y + 9 = 0$.

194. Show that the polars of the point $(1, -2)$ w.r.t. the circle $x^2 + y^2 + 6y + 5 = 0$ and $x^2 + y^2 + 2x + 8y + 5 = 0$ coincide. Prove that there is another point, the polars of which w.r.t. these circles are the same and find its coordinates.

195. Let $C$ be the center of a circle. The lines $L_1$ and $L_2$ are the polars of the points $A$ and $B$ respectively. w.r.t. the circle. Perpendiculars $AM$ and $BN$ are dropped from $A$ to the line $L_2$ and from $B$ to $L_1$. Prove that $CA : CB = AM : BN$.

196. Find the polar of the point of intersection of the line $4x - y = 11$ and $x - 2y = 1$ w.r.t. the circle $x^2 + y^2 = 7$.

197. Find the polar of the point $(4, -1)$ w.r.t. the circle $2x^2 + 2y^2 = 11$.

198. Find the polar of the point $(1, -5)$ w.r.t. the circle $x^2 + y^2 - 8x + 6y + 4 = 0$.

199. Prove that the polar of the point $(p, q)$ w.r.t. the circle $x^2 + y^2 = a^2$ touches $(x - c)^2 + (y - d)^2 = b^2$ if $b^2(p^2 + q^2) = (a^2 - cp - dq)^2$.

200. Show that the polar of the origin w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ touches the circle $x^2 + y^2 = a^2$ if $c^2 = a^2(f^2 + g^2)$.

201. Prove that if the pole of a straight line w.r.t. the circle $x^2 + y^2 = c^2$ lies on the circle $x^2 + y^2 = 9c^2$, the line is a tangent to the circle $9x^2 + 9y^2 = c^2$.

202. Find the pole of the straight line $9x + y - 28 = 0$ w.r.t. the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$.

203. Find the pole of the straight line $2x - y + 10 = 0$ w.r.t. the circle $x^2 + y^2 - 7x + 5y - 1 = 0$.

204. Find the pole of the line $ax + by + c = 0$ w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

205. Prove that the polar of any point w.r.t. a circle is perpendicular to the line joining the point and center of the circle.

206. Prove that the polar of a given point w.r.t. any of the circles $x^2 + y^2 + 2px + c = 0$, where $p$ is a variable, always passes through a fixed point.

207. If the polar of the point $(\alpha, \beta)$ w.r.t. the circle $x^2 + y^2 = a^2$ touches the circle $(x - a)^2 + y^2 = a^2$, show thta $(\alpha, \beta)$ is on the curve given by $y^2 + 2ax = a^2$.

208. Verify that the three points $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ will be collinear if and only if their polars w.r.t. the circle $x^2 + y^2 = a^2$ are concurrent.

209. Show that the circles $x^2 + y^2 - 2x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 6 = 0$ cut each other orthogonally.

210. If $S = 0$ and $S_1 = 0$ are the two circles with radii $a$ and $a_1$ respectively. Show that the circles $\frac{S}{a} \pm \frac{S_1}{a} = 0$ intersect at right angles.

211. Prove that the two circles pass through the points $(0, 0)$ and $(0, -a)$ and touch the line $y = mx + c$ will cut orthogonally if $c^2 = a^2(2 + m^2)$.

212. Obtain the equation of the circle orthogonal to both the circles $x^2 + y^2 + 3x - 5y + 6 = 0$ and $4x^2 + 4y^2 - 28x + 29 = 0$, and whose center lies on the line $3x + 4y + 1 = 0$.

213. Prove that a circle cutting the circle $x^2 + y^2 = 4$ orthogonally and having its center on the line $2x - 2y + 9 = 0$, passes through two fixed points, and find the points.

214. Prove that the general equation of circles cutting the circles $x^2 + y^2 + 2g_r x + 2f_r y + c_r = 0; r = 1, 2$, orthogonally is $\begin{vmatrix} x^2+y^2 & -x & -y \\ c_1 & g_1 & f_1 \\ c_2 & g_2 & f_2 \end{vmatrix} + k \begin{vmatrix} -x & -y & 1 \\ g_1 & f_1 & 1 \\ g_2 & f_2 & 1 \end{vmatrix} = 0.$

215. For what value of $k$ the circles $x^2 + y^2 + 5x + 3y + 7 = 0$ and $x^2 + y^2 - 8x + 6y + k = 0$ cut orthogonally.

216. Find the equation of the circle passing through the origin and cutting the circles $x^2 + y^2 - 4x + 6y + 10 = 0$ and $x^2 + y^2 + 12y + 6 = 0$ at right angles.

217. Find the equation of the circle passing through the origin and has its center on the line $x + y + 4 = 0$ and cuts the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.

218. If two circles cut a third circle orthogonally then prove that their common chord will pass through the center of the third circle.

219. If a circle cuts orthogonally three circles $S_1 = 0, S_2 = 0, S_3 = 0$; prove that it cuts orthogonally any circle of the form $kS_1 + lS_2 + mS_3 = 0$.

220. Prove that the two circles each of which passes through the points $(0, k)$ and $(0, -k)$ and touches the line $y = mx + b$ will cut orthogonally if $b^2 = k^2(2 + m^2)$.

221. Find the general equation of a circle cutting $x^2 + y^2 = c^2$ orthogonally and show that if it passes through the point $(a, b)$ will also pass through the point $\left(\frac{c^2 a}{a^2+b^2}, \frac{c^2 b}{a^2+b^2}\right)$.

222. If $P$ and $Q$ be a pair of conjugate points w.r.t. a circle $S$, prove that the circle on $PQ$ as a diameter cuts the circle $S$ orthogonally.

223. Prove that the circle $x^2 + y^2 - 6x - 4y + 9 = 0$ bisects the circumference of the circle $x^2 + y^2 - 8x - 6y + 23 = 0$.

224. Find the equation of a circle which is coaxal with the circles $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ and $x^2 + y^2 + 4x + 2y + 1 = 0$. It is given that the center of the circle to be determined lies on the radical axis of these circles.

225. If the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2x^2 + 2y^2 + 3x + 8y + 2c = 0$ touches the circle $x^2 + y^2 + 2x - 2y + 1 = 0$, show that either $g = \frac{3}{4}$ or $f = 2$.

226. Find the general equation of circles, any two of which have the same radical axis as that of the circles $x^2 + y^2 + 2x + 4y - 6 = 0$ and $x^2 + y^2 = 4$.

227. The equations of three circles are $x^2 + y^2 = 1, x^2 + y^2 - 8x + 15 = 0. x^2 + y^2 + 10y + 24 = 0$. Determine the coordinates of the point such that the tangents drawn from it to the three circles are equal in length.

228. The polars of a point $P$ w.r.t. two given circles meet in a point $Q$; show that the radical axis of the circles bisedct the line $PQ$.

229. Show that the locus of a point such that the ratio of its distances from two given points is a constant, is a circle. Hence, show that this circle cannot pass through the given points.

230. Two rods of lengths $a$ and $b$ slide along the axes in a manner that their ends are always concyclic. Find the locus of the center of the circle passing through these ends.

231. Two straight lines rotate about two fixed points. If they start from their positions of coincidence such that one rotates at the rate double that of the other. Prove that locus of their point of intersection is a circle.

232. A circle of radius $r$ passes through the origin $O$, and cuts the axes at $A$ and $B$. Let $P$ be the foot of the perpendicular from the origin to the line $AB$. Find the equation of the locus of $P$.

233. Show that the locus of points from which the tangents drawn to a circle are orthogonal, is a concentric circle or find the equation of the cirector circle of the circle $x^2 + y^2 = a^2$.

234. Find the locus of the point of the point of intersection of tangents to the circle $x = a\cos\theta, y = a\sin\theta$ at points whose parametric angles differ by $\frac{\pi}{3}$.

235. The circles $x^2 + y^2 + 2ax - c^2 = 0$ and $x^2 + y^2 + 2bx - c^2 = 0$ intersect at $A$ and $B$. A line through $A$ meets one circle at $P$ and a parallel line through $B$ meets at the other circle at $Q$. Show that the locus of the mid-point of $PQ$ is a circle.

236. Find the condition that the chord of contact of tangents from the point $(\alpha, \beta)$ to the circle $x^2 + y^2 = a^2$ should subtend a right angle at the center. Hence, find the locus of $(\alpha, \beta)$.

237. A tangent is drawn to each of the circle $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$. Show that if the two tangents are mutually perpendicular, the locus of their point of intersection is a cricle concentric with the given circle.

238. Show that the locus of the point, the tangents from which to the circle $x^2 + y^2 = a^2$ include a constant angle $\alpha$ is $\left(x^2 + y^2 - 2a^2\right)^2 \tan^2\alpha = 4a^2(x^2 + y^2 - a^2)$.

239. A straight line passes through the fixed point $(h, k)$. Find the locus of the foot of the perpendicular drawn to it from the origin.

240. $O$ is a fixed point and $AP$ and $BQ$ are two fixed parallel straight lines; $BOA$ is perpendicular to both and $\angle POQ$ is a right angle. Prove that the locus of the foot of the perpendicular drawn from $O$ upon $PQ$ is the circle on $AB$ as diameter.

241. A variable circle passes through the point $P(1, 2)$ and touches the $x$-axis; show that the locus of the other end of the diameter through $P$ is $(x - 1)^2 = 8y$.

242. Find the locus of a point, which is such that the lengths of the tangents from it to two concentric circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ vary inversely as their radii.

243. A point moves such that the sum of squares of its distances from the sides of a square of side unity is equal to 9. Show that the locus of a circle whose center is coincides with the center of the square. Also, find its radius.

244. Find the locus of the center of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ when the pair of tangents drawn from the origin to the circle are perpendicular to each other.
245. Determine the locus of centers of the circle which touches the two circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = 4ax$ externally.
246. Find the locus of the centers of the circle which cut the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ and $x^2 + y^2 - 4x + 6y + 4 = 0$ orthogonally is $8x - 12y + 5 = 0$.
247. Find the locus of the foot of the perpendicular drawn from a fixed point on the $x$-axis to any tangent to the circle $x^2 + y^2 = a^2$.
248. The tangent at any point $P$ on the circle $x^2 + y^2 = 2$ cuts the axes in $L$ and $M$. Find the locus of the middle point of $LM$.
249. A triangle has two of its sides along the axes, its third side touches the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. Find the equation of the locus of the circumcenter of the triangle.
250. The point $A(1, 5)$ is joined to any point $P$ of the circle $x^2 + y^2 = 4$. Find the locus of the middle point of $AP$ as $P$ moves on the circle.
251. Chords of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ drawn through a fixed point $A(a, b)$. Find the locus of the mid-points of these chords and interpret the locus.
252. A straight line moves so that the algebraic sum of the perpendiculars drawn to it from two fixed points is constant, show that the line always touches a fixed circle.

# II

# Answers

# 1 Answers of Coordinates

1. We take the two perpendicular lines as axes of the coordinates. Let $(x, y)$ be any point satisfying the given condition. According to condition $x + y = a$.
2. The area of a triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ is given by $\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

   Substituting the given points, we compute $\Delta = \frac{1}{2}|1(6 - (-1)) + (-7)((-1) - 3) + 5(3 - 6)|$

   This simplifies to $\Delta = \frac{1}{2}|7 + 28 - 15|$

   Thus, $\Delta = \frac{1}{2}|20| = 10$.
3. Substituting the given points in the formula, we obtain $\Delta = \frac{1}{2}|0(6 - (-2)) + 3((-2) - 4) + (-8)(4 - 6)|$

   This simplifies to $\Delta = \frac{1}{2}|-2| = 1$.
4. Substituting the given coordinates, we get $\Delta = \frac{1}{2}.\ |5((-3) - (-5)) + (-9)((-5) - 2) + (-3)(2 - (-3))|$

   Simplifying each term, $= \frac{1}{2}.\ |5(2) + (-9)(-7) + (-3)(5)|$

   This gives $\Delta = \frac{1}{2}.\ |10 + 63 - 15|$

   So, $\Delta = \frac{1}{2}.\ |58| = 29$.
5. $\Delta = \frac{1}{2}|a((b - c) - c) + a(c - (b + c)) + (-a)((b + c) - (b - c))|$

   Simplifying each term, $\Delta = \frac{1}{2}|a(b - 2c) + a(-b) + (-a)(2c)|$

   Combining like terms, $\Delta = \frac{1}{2}|ab - 2ac - ab - 2ac| = \frac{1}{2}| - 4ac|$.
6. $\Delta = \frac{1}{2}|a \cos\varphi_1(a \sin\varphi_2 - a \sin\varphi_3) + a \cos\varphi_2(a \sin\varphi_3 - a \sin\varphi_1) + a \cos\varphi_3(a \sin\varphi_1 - a \sin\varphi_2)|$

   Factor out $a^2$, $\Delta = \frac{a^2}{2}|\cos\varphi_1(\sin\varphi_2 - \sin\varphi_3) + \cos\varphi_2(\sin\varphi_3 - \sin\varphi_1) + \cos\varphi_3(\sin\varphi_1 - \sin\varphi_2)|$

   $\Delta = \frac{a^2}{2}|\sin(\varphi_2 - \varphi_1) + \sin(\varphi_3 - \varphi_2) + \sin(\varphi_1 - \varphi_3)|$.
7. $\Delta = \frac{1}{2}|am_1^2(2am_2 - 2am_3) + am_2^2(2am_3 - 2am_1) + am_3^2(2am_1 - 2am_2)|$

   $\Delta = a^2|m_1^2m_2 - m_1^2m_3 + m_2^2m_3 - m_2^2m_1 + m_3^2m_1 - m_3^2m_2|$

   $\Delta = a^2|(m_1 - m_2)(m_2 - m_3)(m_3 - m_1)|$.
8. $\Delta = \frac{1}{2}\ |1((-2) - 16) + 3(16 - 4) + (-3)(4 - (-2))|$

   $\Delta = \frac{1}{2}| - 18 + 36 - 18|$

   $\Delta = \frac{1}{2}\ |0| = 0$

Since the area is zero, the points $A, B$, and $C$ lie on a straight line and are collinear.

9. $\Delta = \frac{1}{2}|(-\frac{1}{2})(6-8) + (-5)(8-3) + (-8)(3-6)|$

   $\Delta = \frac{1}{2}|(-\frac{1}{2})(-2) + (-5)(5) + (-8)(-3)|$

   $\Delta = \frac{1}{2}\ |1 - 25 + 24| = 0$

   Since the area of the triangle is zero, therefore, the points are collinear.

10. $\Delta = \frac{1}{2}\ |a((c+a)-(a+b)) + b((a+b)-(b+c)) + c((b+c)-(c+a))|$

    $\Delta = \frac{1}{2}\ |a(c+a-a-b) + b(a+b-b-c) + c(b+c-c-a)|$

    $\Delta = \frac{1}{2}\ |a(c-b) + b(a-c) + c(b-a)|$

    $\Delta = \frac{1}{2}\ |ac - ab + ab - bc + bc - ac| = \frac{1}{2}\ |0|$

    Hence, the area is zero, which proves that the points collinear.

11. Let $ABC$ be a triangle with vertices $A, B$, and $C$. Let $D$ be the midpoint of $BC$. Then the coordinates of $D$ are $\left(\frac{xB+xC}{2}, \frac{yB+yC}{2}\right)$.

    $AB^2 = (xB - xA)^2 + (yB - yA)^2$, and $AC^2 = (xC - xA)^2 + (yC - yA)^2$

    $CD^2 = \left(xC - \frac{xB+xC}{2}\right)^2 + \left(yC - \frac{yB+yC}{2}\right)^2 = \left(\frac{xC-xB}{2}\right)^2 + \left(\frac{yC-yB}{2}\right)^2 = \frac{1}{4}BC^2$

    Now, consider the sum $AB^2 + AC^2$. Using the midpoint formula, we can write $AC^2 + AD^2 + CD^2 = AB^2 + \frac{AC^2}{2}$, and after simplification, we obtain $AB^2 + AC^2 = 2(AD^2 + CD^2)$

    Hence, in any triangle $ABC$, if $D$ is the midpoint of $BC$, then $AB^2 + AC^2 = 2(AD^2 + CD^2)$.

12. Let triangle $ABC$ have vertices $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$.

    Let $D$, $E$, and $F$ be the midpoints of $BC$, $CA$, and $AB$ respectively. Then $D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$, $E = \left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right)$, $F = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

    Let $P$ be the point that divides $AD$ internally in the ratio $2:1$. By the section formula, $P = \left(\frac{1.x_1+2.\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1.y_1+2.\left(\frac{y_2+y_3}{2}\right)}{1+2}\right)$.

    Simplifying, $P = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

    Similarly, for the line $BE$, let $Q$ divide it internally in the ratio $2:1$. Then $Q = \left(\frac{1.x_2+2.\left(\frac{x_3+x_1}{2}\right)}{1+2}, \frac{1.y_2+2.\left(\frac{y_3+y_1}{2}\right)}{1+2}\right)$,

    which simplifies to $Q = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

    For the line $CF$, let $R$ divide it internally in the ratio $2:1$. Then $R = \left(\frac{1.x_3+2.\left(\frac{x_1+x_2}{2}\right)}{1+2}, \frac{1.y_3+2.\left(\frac{y_1+y_2}{2}\right)}{1+2}\right)$,

    which simplifies to $R = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

    Hence, the point that divides $AD$ in the ratio $2:1$ also divides $BE$ and $CF$ in the same ratio. This point is the centroid of triangle $ABC$.

13. Let the vertices of the quadrilateral be $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, and $D(x_4, y_4)$.

    The area of a quadrilateral can be computed by dividing it into two triangles, for example $ABC$ and $ACD$, and adding their areas.

    The area of triangle $ABC$ is $\frac{1}{2} * |x_1 * (y_2 - y_3) + x_2 * (y_3 - y_1) + x_3 * (y_1 - y_2)|$.

    The area of triangle $ACD$ is $\frac{1}{2} * |x_1 * (y_3 - y_4) + x_3 * (y_4 - y_1) + x_4 * (y_1 - y_3)|$.

    Hence, the area of the quadrilateral is

    $\square ABCD = \frac{1}{2} * |x_1 * (y_2 - y_3) + x_2 * (y_3 - y_1) + x_3 * (y_1 - y_2)| + \frac{1}{2} * |x_1 * (y_3 - y_4) + x_3 * (y_4 - y_1) + x_4 * (y_1 - y_3)|$.

14. Substituting the given coordinates in the formula obtained in previous problem, we get

    $\Delta = \frac{1}{2} * |1 * 4 + 3 * (-2) + 5 * (-7) + 4 * 1 - (1 * 3 + 4 * 5 + (-2) * 4 + (-7) * 1)|$

    Simplifying each term, we get $\Delta = \frac{1}{2} * |4 - 6 - 35 + 4 - (3 + 20 - 8 - 7)|$

    $\Delta = \frac{1}{2} * |-33 - (-12)| = \frac{1}{2} * |-33 + 12| = \frac{1}{2} * |-21| = \frac{21}{2} = 10.5$.

15. Like previous problem, $\Delta = \frac{1}{2} * |(-1) * (-9) + (-3) * 8 + 5 * 9 + 3 * 0 - (0 * (-3) + (-9) * 5 + 8 * 3 + 9 * (-1))|$

    Simplifying each term, we get $\Delta = \frac{1}{2}\ |9 - 24 + 45 + 0 - (0 - 45 + 24 - 9)|$

    $\Delta = \frac{1}{2}\ |30 - (-30)| = \frac{1}{2}\ |30 + 30| = \frac{1}{2}60 = 30$.

16. The distance between two points in polar coordinates can be found by first converting to Cartesian coordinates: $P_1 = (r_1 \cos\theta_1, r_1 \sin\theta_1)$, $P_2 = (r_2 \cos\theta_2, r_2 \sin\theta_2)$

    The distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

    Substituting the Cartesian coordinates, we get $L = \sqrt{(r_2 \cos\theta_2 - r_1 \cos\theta_1)^2 + (r_2 \sin\theta_2 - r_1 \sin\theta_1)^2}$

    Expanding the squares, we have $L =$

    $\sqrt{r_2^2 \cos^2\theta_2 - 2r_1r_2 \cos\theta_1 \cos\theta_2 + r_1^2 \cos^2\theta_1 + r_2^2 \sin^2\theta_2 - 2r_1r_2 \sin\theta_1 \sin\theta_2 + r_1^2 \sin^2\theta_1}$.

    Grouping terms, we get

    $L = \sqrt{r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2) - 2r_1r_2(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)}$

    Using $\cos^2\theta + \sin^2\theta = 1$ and $\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 = \cos(\theta_1 - \theta_2)$, we get $L = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}$.

17. The distance between two points in polar coordinates is $L = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}$

    Substituting the given values, we get $L = \sqrt{2^2 + 4^2 - 224 \cos(30^\circ - 120^\circ)}$

    Simplifying inside the cosine, $L = \sqrt{4 + 16 - 16 \cos(-90^\circ)}$.

Since $\cos(-90^\circ) = 0$, we have $L = \sqrt{4 + 16 - 160} = \sqrt{20}$.

18. Like previous problem $L = \sqrt{(-3)^2 + 7^2 - 2(-3)7\cos(45^\circ - 105^\circ)}$.

    Simplifying, $L = \sqrt{9 + 49 - 2(-3)7\cos(-60^\circ)}$.

    We know $\cos(-60^\circ) = \frac{1}{2}$, so $L = \sqrt{58 - 2(-3)7\frac{1}{2}}$.

    Simplifying the middle term, $2(-3)7\frac{1}{2} = -21$, so $L = \sqrt{58 - (-21)} = \sqrt{58 + 21} = \sqrt{79}$.

    Hence, the distance between the points is $L = \sqrt{79}$.

19. Like previous problem $L = \sqrt{a^2 + (3a)^2 - 2a3a\cos(\frac{\pi}{2} - \frac{\pi}{6})}$.

    Simplifying, $L = \sqrt{a^2 + 9a^2 - 6a^2\cos(\frac{\pi}{3})}$.

    Since $\cos(\frac{\pi}{3}) = \frac{1}{2}$, we have $L = \sqrt{10a^2 - 3a^2}$.

    $L = \sqrt{7a^2} = a\sqrt{7}$.

20. Let the vertices of the triangle be $P_1(r_1, \theta_1)$, $P_2(r_2, \theta_2)$, and $P_3(r_3, \theta_3)$.

    First, convert the polar coordinates to Cartesian coordinates: $P_1 = (r_1\cos(\theta_1), r_1\sin(\theta_1))$, $P_2 = (r_2\cos(\theta_2), r_2\sin(\theta_2))$, $P_3 = (r_3\cos(\theta_3), r_3\sin(\theta_3))$.

    The area of a triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is $\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$.

    Substituting the Cartesian coordinates in terms of polar coordinates, we get $\Delta = \frac{1}{2}|r_1\cos(\theta_1)(r_2\sin(\theta_2) - r_3\sin(\theta_3)) + r_2\cos(\theta_2)(r_3\sin(\theta_3) - r_1\sin(\theta_1)) + r_3\cos(\theta_3)(r_1\sin(\theta_1) - r_2\sin(\theta_2))|$.

    Hence, the area of the triangle is $\Delta = \frac{1}{2}|r_1(r_2\sin(\theta_2) - r_3\sin(\theta_3))\cos(\theta_1) + r_2(r_3\sin(\theta_3) - r_1\sin(\theta_1))\cos(\theta_2) + r_3(r_1\sin(\theta_1) - r_2\sin(\theta_2))\cos(\theta_3)|$.

21. Let the vertices be $P_1(1, 30^\circ)$, $P_2(2, 60^\circ)$, $P_3(3, 90^\circ)$

    Convert to Cartesian: $P_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $P_2 = \left(1, \sqrt{3}\right)$, $P_3 = (0, 3)$

    Area formula: $\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

    Substitute: $\Delta = \frac{1}{2}\left|\left(\frac{\sqrt{3}}{2}\right)\left(\sqrt{3} - 3\right) + 1\left(3 - \frac{1}{2}\right) + 0\right| = \frac{1}{2}\left|4 - 3\frac{\sqrt{3}}{2}\right| = 2 - 3\frac{\sqrt{3}}{4}$

    $= 2 - 3\frac{\sqrt{3}}{4}$.

22. Let the vertices be $P_1(-3, 30^\circ)$, $P_2(5, 150^\circ)$, $P_3(7, 210^\circ)$.

    Convert to Cartesian coordinates: $P_1 = (-3\cos(30^\circ), -3\sin(30^\circ)) = \left(-3\frac{\sqrt{3}}{2}, -\frac{3}{2}\right)$, $P_2 = (5\cos(150^\circ), 5\sin(150^\circ)) = \left(-5\frac{\sqrt{3}}{2}, \frac{5}{2}\right)$, $P_3 = (7\cos(210^\circ), 7\sin(210^\circ)) = \left(-7\frac{\sqrt{3}}{2}, -\frac{7}{2}\right)$.

    Substitute the coordinates: $\Delta = \frac{1}{2}\left|\left(-3\frac{\sqrt{3}}{2}\right)\left(\frac{5}{2} - \left(-\frac{7}{2}\right)\right) + \left(-5\frac{\sqrt{3}}{2}\right)\left(\left(-\frac{7}{2}\right) - \left(-\frac{3}{2}\right)\right) + \left(-7\frac{\sqrt{3}}{2}\right)\left(\left(-\frac{3}{2}\right) - \frac{5}{2}\right)\right|$

$\Delta = \frac{1}{2}\left[-9\sqrt{3} + 5\sqrt{3} + 14\sqrt{3}\right] = 5\sqrt{3}.$

23. Let the vertices be $P_1\left(-a, \frac{\pi}{6}\right)$, $P_2\left(a, \frac{\pi}{2}\right)$, $P_3\left(-2a, 2\frac{\pi}{3}\right)$.

    Convert to Cartesian coordinates: $P_1 = \left(-a\cos\left(\frac{\pi}{6}\right), -a\sin\left(\frac{\pi}{6}\right)\right) = \left(-a\frac{\sqrt{3}}{2}, -\frac{a}{2}\right)$, $P_2 = \left(a\cos\left(\frac{\pi}{2}\right), a\sin\left(\frac{\pi}{2}\right)\right) = (0, a)$, $P_3 = \left(-2a\cos\left(2\frac{\pi}{3}\right), -2a\sin\left(2\frac{\pi}{3}\right)\right) = \left(a, -a\sqrt{3}\right)$.

    Substitute the coordinates: $\Delta = \frac{1}{2}\left|\left(-a\frac{\sqrt{3}}{2}\right)\left(a - \left(-a\sqrt{3}\right)\right) + 0\left(\left(-a\sqrt{3} - \left(-\frac{a}{2}\right)\right)\right) + (a)\left(\left(-\frac{a}{2}\right) - a\right)\right|$

    $\Delta = \frac{1}{2}| - \left(a^2\frac{\sqrt{3}+3}{2}\right) + 0 - 3\frac{a^2}{2}| = \frac{1}{2}| - a^2\frac{\sqrt{3}+6}{2}| = a^2\frac{\sqrt{3}+6}{4}.$

24. Let the points be $P_1(a\cos\alpha, a\sin\alpha)$ and $P_2(a\cos\beta, a\sin\beta)$.

    The distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is $L = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$.

    Substitute the coordinates: $L = \sqrt{(a\cos\beta - a\cos\alpha)^2 + (a\sin\beta - a\sin\alpha)^2}$.

    Factor out $a$ in each term: $L = \sqrt{a^2(\cos\beta - \cos\alpha)^2 + a^2(\sin\beta - \sin\alpha)^2}$.

    $L = \sqrt{a^2((\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2)}$.

    Take $a$ out of the square root: $L = a\sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2}$.

    Use the identity $(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 = 2 - 2\cos(\beta - \alpha)$: $L = a\sqrt{2 - 2\cos(\beta - \alpha)}$.

    Factor 2: $L = a\sqrt{2(1 - \cos(\beta - \alpha))} = a\sqrt{2}\sqrt{1 - \cos(\beta - \alpha)}$.

25. Let the points be $P_1\left(at_1^2, 2at_1\right)$ and $P_2\left(at_2^2, 2at_2\right)$.

    The distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is $L = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$.

    Substitute the coordinates: $L = \sqrt{\left(at_2^2 - at_1^2\right)^2 + \left(2at_2 - 2at_1\right)^2}$.

    Factor out $a$ from the first term and 2 a from the second term: $L = \sqrt{a^2\left(t_2^2 - t_1^2\right)^2 + \left(2a\left(t_2 - t_1\right)\right)^2}$.

    Simplify: $\left(t_2^2 - t_1^2\right)^2 = \left(t_2 - t_1\right)^2\left(t_2 + t_1\right)^2$, $\left(2a\left(t_2 - t_1\right)\right)^2 = 4a^2\left(t_2 - t_1\right)^2$.

    So $L = \sqrt{a^2\left(t_2 - t_1\right)^2\left(t_2 + t_1\right)^2 + 4a^2\left(t_2 - t_1\right)^2}$.

    Factor $a^2\left(t_2 - t_1\right)^2$: $L = \sqrt{a^2\left(t_2 - t_1\right)^2\left(\left(t_2 + t_1\right)^2 + 4\right)}$.

    Take $a\left(t_2 - t_1\right)$ out of the square root: $L = a\left|t_2 - t_1\right|\sqrt{\left(t_2 + t_1\right)^2 + 4}$.

26. The equation $x^2 + y^2 = a^2$ can be rewritten in polar coordinates. Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Substituting, we get:

    $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2$

Hence, the equation becomes $r^2 = a^2$, or equivalently:

$r = a$

27. Consider the equation $y = x \tan \alpha$. In polar coordinates, we have $x = r \cos \theta$ and $y = r \sin \theta$.

    Substituting, we get: $r \sin \theta = r \cos \theta \tan \alpha$

    Dividing both sides by $r \cos \theta$ (assuming $r \neq 0$ and $\cos \theta \neq 0$), we obtain: $\tan \theta = \tan \alpha$

    Hence, in polar coordinates, the equation becomes: $\theta = \alpha + n\pi$, where $n$ is any integer.

28. Consider the equation $x^2 + y^2 = 2ax$. In polar coordinates, $x = r \cos \theta$ and $y = r \sin \theta$.

    Substituting these, we get: $r^2 \cos^2 \theta + r^2 \sin^2 \theta = 2ar \cos \theta$

    Simplifying the left side: $r^2(\cos^2 \theta + \sin^2 \theta) = r^2 = 2ar \cos \theta$

    Dividing both sides by $r$ (assuming $r \neq 0$), we obtain: $r = 2a \cos \theta$

    Thus, in polar coordinates, the equation becomes: $r = 2a \cos \theta$.

29. Consider the equation $x^2 - y^2 = 2ay$. In polar coordinates, $x = r \cos \theta$ and $y = r \sin \theta$.

    Substituting these, we get: $r^2 \cos^2 \theta - r^2 \sin^2 \theta = 2ar \sin \theta$

    Factor $r^2$ on the left: $r^2(\cos^2 \theta - \sin^2 \theta) = 2ar \sin \theta$

    Divide both sides by $r$ (assuming $r \neq 0$): $r(\cos^2 \theta - \sin^2 \theta) = 2a \sin \theta$

    Hence, in polar coordinates, the equation becomes: $r = 2a \sin \frac{\theta}{\cos^2 \theta - \sin^2 \theta}$.

30. Consider the equation $x^2 = y^2(2a - x)$. In polar coordinates, $x = r \cos \theta$ and $y = r \sin \theta$.

    Substituting these, we get: $(r \cos \theta)^2 = (r \sin \theta)^2(2a - r \cos \theta)$

    Simplifying both sides: $r^2 \cos^2 \theta = r^2 \sin^2 \theta(2a - r \cos \theta)$

    Divide both sides by $r^2$ (assuming $r \neq 0$): $\cos^2 \theta = \sin^2 \theta(2a - r \cos \theta)$

    Solving for $r$: $r = \frac{2a \sin^2 \theta - \cos^2 \theta}{\sin^2 \theta \cos \theta}$.

31. Consider the equation $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

    Using $x = r \cos \theta$ and $y = r \sin \theta$, we have: $(r^2)^2 = a^2 r^2(\cos^2 \theta - \sin^2 \theta)$

    Dividing both sides by $r^2$: $r^2 = a^2(\cos^2 \theta - \sin^2 \theta)$.

32. Given the polar equation $r = a$, we square both sides to obtain $r^2 = a^2$.

    Since $r^2 = x^2 + y^2$, the Cartesian form is: $x^2 + y^2 = a^2$.

33. Given the polar equation $\theta = \tan^{-1} m$, taking the tangent gives: $\tan \theta = m$

    Since $\tan \theta = \frac{y}{x}$, we obtain the Cartesian equation: $y = mx$.

34. Given the polar equation $r = a \cos \theta$, multiply both sides by $r$: $r^2 = ar \cos \theta$

Using $r^2 = x^2 + y^2$ and $r\cos\theta = x$, we obtain: $x^2 + y^2 = ax$.

35. Given $r^2 = a^2 * \sin(2 * \theta)$

Using $x = r * \cos(\theta)$, $y = r * \sin(\theta)$, and $r^2 = x^2 + y^2$, $\sin(2 * \theta) = 2 * \sin(\theta) * \cos(\theta)$

Substituting: $x^2 + y^2 = a^2 * 2 * \left(\frac{y}{\sqrt{x^2+y^2}}\right) * \left(\frac{x}{\sqrt{x^2+y^2}}\right)$

$x^2 + y^2 = 2 * a^2 * x * \frac{y}{x^2+y^2}$

Multiplying both sides by $x^2 + y^2$: $\left(x^2 + y^2\right)^2 = 2 * a^2 * x * y$.

36. Given the polar equation $r^2 \sin 2\theta = 2a^2$, use the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

Since $r\sin\theta = y$ and $r\cos\theta = x$, we have: $2xy = 2a^2$

Dividing both sides by 2 gives the Cartesian equation: $xy = a^2$.

37. Given the polar equation $\sqrt{r}\cos\left(\frac{\theta}{2}\right) = \sqrt{a}$, square both sides: $r\cos^2\left(\frac{\theta}{2}\right) = a$

Using $\cos^2\left(\frac{\theta}{2}\right) = \frac{1+\cos\theta}{2}$, we get: $r(1 + \cos\theta) = 2a$

Since $\cos\theta = \frac{x}{r}$, this gives: $r + x = 2a$

With $r = \sqrt{x^2 + y^2}$, the Cartesian equation is: $x^2 + y^2 = (2a - x)^2$.

38. Given the polar equation $\sqrt{r} = \sqrt{a}\sin\left(\frac{\theta}{2}\right)$, squaring gives: $r = a\sin^2\left(\frac{\theta}{2}\right)$

Using $\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{2}$, we obtain: $r = a\frac{1-\cos\theta}{2}$

Substituting $\cos\theta = \frac{x}{r}$ and $r^2 = x^2 + y^2$ leads to: $\left(x^2 + y^2 + a\frac{x}{2}\right)^2 = a^2\frac{x^2+y^2}{4}$.

39. The polar equation $r(\cos 3\theta + \sin 3\theta) = 5k\sin\theta\cos\theta$ can be converted to Cartesian coordinates using $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$.

Using the triple-angle formulas $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, we can write:
$\cos 3\theta = \frac{4x^3-3xr^2}{r^3}$ and $\sin 3\theta = \frac{3yr^2-4y^3}{r^3}$

Adding these gives: $\cos 3\theta + \sin 3\theta = \frac{4(x^3-y^3)+3r^2(y-x)}{r^3}$

Multiplying both sides of the original equation by $r$: $r(\cos 3\theta + \sin 3\theta) = \frac{4(x^3-y^3)+3r^2(y-x)}{r^2}$

The right-hand side $5k\sin\theta\cos\theta$ becomes $5kx\frac{y}{r^2}$

Multiplying both sides by $r^2$ to eliminate the denominator: $4(x^3 - y^3) + 3r^2(y - x) = 5kxy$

Substituting $r^2 = x^2 + y^2$ yields the final Cartesian equation: $4(x^3 - y^3) + 3(x^2 + y^2)(y - x) = 5kxy$

40. Distance formula is $\sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2} = d^2$

Substituting given coordinate in the formula: $\texttt{sqrt}\{(a - 3)^2 + (2 - 4)^2\} = 8^2$

Simplifying gives us $a = 3 + 2\sqrt{15}$ (rejecting negative value for distance).

41. Like previous problem substituting the coordinates gives us $\sqrt{(a + r\cos\theta - a)^2 + (b + r\sin\theta - b)^2} = r^2$, which is independent of $\theta$.

42. Let the points be $A = (\csc^2(\theta), 0)$, $B = (0, \sec^2(\theta))$, and $C = (1, 1)$.

    The distance formula between two points $(x_1, y_1)$ and $(x_2, y_2)$ is: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

    Compute the distance $AB$: $AB = \sqrt{(0 - \csc^2(\theta))^2 + (\sec^2(\theta) - 0)^2}$

    $= \sqrt{(\csc^2(\theta))^2 + (\sec^2(\theta))^2} = \sqrt{\csc^4(\theta) + \sec^4(\theta)}$

    Compute the distance $BC$:

    $BC = \sqrt{(1 - 0)^2 + (1 - \sec^2(\theta))^2} = \sqrt{1 + (1 - \sec^2(\theta))^2}$

    Compute the distance $AC$:

    $AC = \sqrt{(1 - \csc^2(\theta))^2 + (1 - 0)^2} = \sqrt{(1 - \csc^2(\theta))^2 + 1}$

    Using the Pythagorean identities: $\csc^2(\theta) = \frac{1}{\sin^2}(\theta)$ and $\sec^2(\theta) = \frac{1}{\cos^2}(\theta)$

    Substitute these into the distances and simplify. After simplification, we find that: $AB + BC = AC$

    Therefore, the points $A$, $B$, and $C$ are collinear.

43. Let $P = (x, y)$, $A = (a + b, b - a)$, and $B = (a - b, a + b)$.

    Since $P$ is equidistant from $A$ and $B$, we have:

    $\sqrt{(x - (a + b))^2 + (y - (b - a))^2} = \sqrt{(x - (a - b))^2 + (y - (a + b))^2}$

    Squaring both sides: $(x - (a + b))^2 + (y - (b - a))^2 = (x - (a - b))^2 + (y - (a + b))^2$

    Expanding both sides: $(x - a - b)^2 + (y - b + a)^2 = (x - a + b)^2 + (y - a - b)^2$

    Expanding each square:

    $(x - a - b)^2 = x^2 - 2x(a + b) + (a + b)^2 \Rightarrow (y - b + a)^2 = y^2 - 2y(b - a) + (b - a)^2$

    $\Rightarrow (x - a + b)^2 = x^2 - 2x(a - b) + (a - b)^2 \Rightarrow (y - a - b)^2 = y^2 - 2y(a + b) + (a + b)^2$

    Adding the terms on each side:

    Left: $x^2 + y^2 - 2x(a + b) - 2y(b - a) + (a + b)^2 + (b - a)^2$

    Right: $x^2 + y^2 - 2x(a - b) - 2y(a + b) + (a - b)^2 + (a + b)^2$

    Subtracting $x^2 + y^2 + (a + b)^2$ from both sides:

    $-2x(a + b) - 2y(b - a) + (b - a)^2 = -2x(a - b) - 2y(a + b) + (a - b)^2$

    Simplifying $(b - a)^2 = (a - b)^2$: $-2x(a + b) - 2y(b - a) = -2x(a - b) - 2y(a + b)$

    Divide both sides by -2: $x(a + b) + y(b - a) = x(a - b) + y(a + b)$

Bringing all terms to left-hand side:$x(a+b-a+b)+y(b-a-a-b)=0 \Rightarrow x(2b)+y(-2a)=0$

Simplifying: $2bx-2ay=0$

Dividing both sides by $b(x+y)$: $\frac{a-b}{a+b}=\frac{x-y}{x+y}$

44. Let the points be $A=(3,4)$, $B=(8,-6)$, and $C=(13,9)$.

    Computing the sides:

    $AB=\sqrt{(8-3)^2+(-6-4)^2}=\sqrt{5^2+(-10)^2}=\sqrt{25+100}=\sqrt{125}$

    $BC=\sqrt{(13-8)^2+(9-(-6))^2}=\sqrt{5^2+15^2}=\sqrt{25+225}=\sqrt{250}$

    $AC=\sqrt{(13-3)^2+(9-4)^2}=\sqrt{10^2+5^2}=\sqrt{100+25}=\sqrt{125}$

    Since $AB^2+AC^2=BC^2$, the triangle ABC is right-angled at $A$.

45. Solution is given below:

    1. Computing the sides using the distance formula:

       $AB=\sqrt{\left(-\sqrt{3}-1\right)^2+\left(\sqrt{3}-1\right)^2}=2\sqrt{2}$

       $BC=\sqrt{\left(-1+\sqrt{3}\right)^2+\left(-1-\sqrt{3}\right)^2}=2\sqrt{2}$

       $AC=\sqrt{(-1-1)^2+(-1-1)^2}=2\sqrt{2}$

       Since $AB=BC=AC$, the triangle is *equilateral.*

    2. Let the points be $A=(0,2)$, $B=(7,0)$, and $C=(2,5)$.

       Computing the sides using the distance formula:

       $AB=\sqrt{(7-0)^2+(0-2)^2}=\sqrt{49+4}=\sqrt{53}$

       $BC=\sqrt{(2-7)^2+(5-0)^2}=\sqrt{(-5)^2+5^2}=\sqrt{25+25}=\sqrt{50}$

       $AC=\sqrt{(2-0)^2+(5-2)^2}=\sqrt{4+9}=\sqrt{13}$

       No combination satisfies the Pythagoras theorem exactly, so the triangle is not right-angled.

       Since all sides are different ($\sqrt{53}$, $\sqrt{50}$, $\sqrt{13}$), the triangle is *scalene.*

    3. Let the points be $A=(-2,5)$, $B=(7,10)$, and $C=(3,-4)$.

       Computing the sides using the distance formula:

       $AB=\sqrt{(7-(-2))^2+(10-5)^2}=\sqrt{9^2+5^2}=\sqrt{81+25}=\sqrt{106}$

       $BC=\sqrt{(3-7)^2+(-4-10)^2}=\sqrt{(-4)^2+(-14)^2}=\sqrt{16+196}=\sqrt{212}$

       $AC=\sqrt{(3-(-2))^2+(-4-5)^2}=\sqrt{5^2+(-9)^2}=\sqrt{25+81}=\sqrt{106}$

       $AB=AC=\sqrt{106}$, $BC=\sqrt{212}$

       Since two sides are equal, the triangle is *isosceles.*

       Check for a right angle using the Pythagoras theorem:

       $AB^2+AC^2=106+106=212=BC^2$

Hence, the triangle is *right-angled isosceles* at angle $A$.

46. Let the point be $P = (a\cos(\alpha), a\sin(\alpha))$ and the origin be $O = (0, 0)$.

    The distance formula between two points $(x_1, y_1)$ and $(x_2, y_2)$ is:

    $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

    Computing the distance $OP$:

    $OP = \sqrt{(a\cos(\alpha) - 0)^2 + (a\sin(\alpha) - 0)^2}$

    Using the Pythagorean identity $\cos(\alpha)^2 + \sin(\alpha)^2 = 1$:

    $OP = \sqrt{a^2 * 1} = \sqrt{a^2} = a$, which is independent of $\alpha$

47. Let the points be $A = (6, -1)$, $B = (1, 3)$, and $C = (x, 8)$.

    Given $AB = BC$

    $AB = \sqrt{(1-6)^2 + (3-(-1))^2} = \sqrt{(-5)^2 + 4^2} = \sqrt{25+16} = \sqrt{41}$

    $BC = \sqrt{(x-1)^2 + (8-3)^2} = \sqrt{(x-1)^2 + 5^2} = \sqrt{(x-1)^2 + 25}$

    $\Rightarrow \sqrt{41} = \sqrt{(x-1)^2 + 25}$

    Squaring both sides: $41 = (x-1)^2 + 25 \Rightarrow (x-1)^2 = 16$

    $\Rightarrow x - 1 = 4$ or $x - 1 = -4 \Rightarrow x = 5$ or $x = -3$

48. Let the points be $A = (1, 5)$, $B = (2, 4)$, and $C = (3, 3)$.

    $AB = \sqrt{(2-1)^2 + (4-5)^2} = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$

    $BC = \sqrt{(3-2)^2 + (3-4)^2} = \sqrt{1^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$

    $AC = \sqrt{(3-1)^2 + (3-5)^2} = \sqrt{2^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8}$

    Check if $AB + BC = AC$:

    $AB + BC = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

    $AC = \sqrt{8} = 2\sqrt{2}$

    Since $AB + BC = AC$, the points $A$, $B$, and $C$ are collinear.

49. Let the points be $A = (2a, 4a)$, $B = (2a, 6a)$, $C = \left(2a + \sqrt{3}a, 5a\right)$.

    $AB = \sqrt{(2a - 2a)^2 + (6a - 4a)^2} = \sqrt{0 + (2a)^2} = \sqrt{4a^2} = 2a$

    $BC = \sqrt{\left(2a + \sqrt{3}a - 2a\right)^2 + (5a - 6a)^2} = \sqrt{\left(\sqrt{3}a\right)^2 + (-a)^2} = \sqrt{3a^2 + a^2} = \sqrt{4a^2} = 2a$

    $AC = \sqrt{\left(2a + \sqrt{3}a - 2a\right)^2 + (5a - 4a)^2} = \sqrt{\left(\sqrt{3}a\right)^2 + (a)^2} = \sqrt{3a^2 + a^2} = \sqrt{4a^2} = 2a$

    Since $AB = BC = AC = 2a$, all sides are equal.

    Hence, the triangle with vertices $A$, $B$, and $C$ is equilateral.

50. Let the points be $O = (0, 0)$, $A = (a, b)$, and $B = (c, d)$.

    $OA = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$

$OB = \sqrt{(c-0)^2 + (d-0)^2} = \sqrt{c^2 + d^2}$

$AB = \sqrt{(c-a)^2 + (d-b)^2} = \sqrt{(c-a)^2 + (d-b)^2}$

By the cosine law, for triangle $OAB$ with angle $\theta$ at $O$:

$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2 OA OB}$

Substituting the distances:

$\cos\theta = \frac{(a^2+b^2)+(c^2+d^2)-((c-a)^2+(d-b)^2)}{2\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$

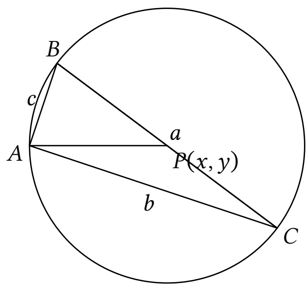
Expanding $AB^2 = (c-a)^2 + (d-b)^2 = c^2 - 2ac + a^2 + d^2 - 2bd + b^2 = a^2 + b^2 + c^2 + d^2 - 2(ac + bd)$

Substituting: $\cos(\theta) = \frac{(a^2+b^2)+(c^2+d^2)-(a^2+b^2+c^2+d^2-2(ac+bd))}{2\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$

$\Rightarrow \cos\theta = \frac{2(ac+bd)}{2\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$

$\Rightarrow \cos\theta = \frac{ac+bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}$.

51. Let the vertices be $A = (-2, -3)$, $B = (-1, 0)$, and $C = (7, -6)$.




Let the circumcenter be $O = (x, y)$, which is equidistant from all three vertices:

$\sqrt{(x+2)^2 + (y+3)^2} = \sqrt{(x+1)^2 + y^2}$

Squaring both sides: $(x+2)^2 + (y+3)^2 = (x+1)^2 + y^2$

$\Rightarrow x^2 + 4x + 4 + y^2 + 6y + 9 = x^2 + 2x + 1 + y^2 \Rightarrow 2x + 6y + 12 = 0 \Rightarrow x + 3y + 6 = 0$

Also, $\sqrt{(x+1)^2 + y^2} = \sqrt{(x-7)^2 + (y+6)^2}$

Squaring both sides: $(x+1)^2 + y^2 = (x-7)^2 + (y+6)^2$

$\Rightarrow x^2 + 2x + 1 + y^2 = x^2 - 14x + 49 + y^2 + 12y + 36$

$\Rightarrow 16x - 12y - 84 = 0 \Rightarrow 4x - 3y - 21 = 0$

Solving the system: From $x + 3y + 6 = 0$, we get $x = -3y - 6$

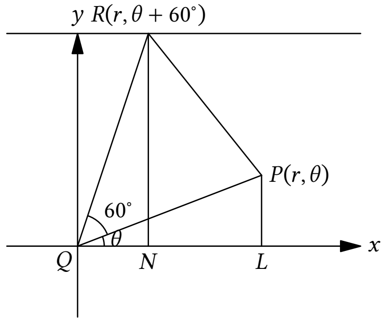
Substitute into $4x - 3y - 21 = 0$: $4(-3y-6) - 3y - 21 = 0 \Rightarrow -15y - 45 = 0 \Rightarrow y = -3$

Then $x = -3(-3) - 6 = 3$

Thus, the circumcenter is $O = (3, -3)$

The circumradius is: $R = \sqrt{(3-(-2))^2 + (-3-(-3))^2} = \sqrt{5^2 + 0^2} = 5$

52. The diagram is given below:




Let $PQ = QR = RP = r$ and $\angle PQX = \theta$ then $\angle RQX = 60^{\circ} + \theta$.

Given $PL = a, RN = 1$.

We take $Q$ as the pole and $QX$ as the initial line. Polar coordinates of $P$ and $R$ will be $(r, \theta)$ and $(r, \theta + 60^{\circ})$, respectively.

Now $r \sin\theta = y$, coordinate of $P = a$ and $r\sin(60^{\circ} + \theta) = y$, coordinate of $R = 1$

$\Rightarrow r(\sin 60^{\circ} \cos\theta + \cos 60^{\circ} \sin\theta) = 1$

$\Rightarrow r\left(\frac{\sqrt{3}}{2}\sqrt{1 - \frac{a^2}{r^2}} + \frac{1}{2}.\frac{a}{r}\right) = 1$

$\Rightarrow r = \frac{2}{\sqrt{3}}\sqrt{a^2 - a + 1}.$

53. Let $A = (3, 4)$ and $C = (1, -1)$ be opposite vertices of the square. Let $B = (x, y)$ and $D = (x', y')$ be the other two vertices.

The center of the square is the midpoint of $AC$: $O = \left(\frac{3+1}{2}, \frac{4+(-1)}{2}\right) = \left(2, \frac{3}{2}\right)$

In a square, the distances from the center to all vertices are equal: $(x-2)^2 + \left(y - \frac{3}{2}\right)^2 = (3-2)^2 + \left(4 - \frac{3}{2}\right)^2 = \frac{29}{4}$

Also, $B$ and $D$ lie on a line through $O$ that is perpendicular to $AC$. Using the fact that diagonals of a square are equal and perpendicular, we solve:

$(x-2)^2 + \left(-\frac{2}{5}x + \frac{23}{10} - \frac{3}{2}\right)^2 = \frac{29}{4}$

Solving this quadratic gives: $x = \frac{9}{2}, y = \frac{1}{2}$ and $x = -\frac{1}{2}, y = \frac{5}{2}$

Hence, the remaining vertices are: $B = \left(\frac{9}{2}, \frac{1}{2}\right), D = \left(-\frac{1}{2}, \frac{5}{2}\right)$.

54. Given $A = (-4, 0)$ and $B = (-1, 4)$, points $C$ and $D$ are symmetric about the $y$ -axis: $C = (4, 0)$ and $D = (1, 4)$

The trapezium is $ABCD$, with vertices in order $A, B, D, C$.

$AB = \sqrt{(-1-(-4))^2 + (4-0)^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$

$BC = \sqrt{(1-(-1))^2 + (4-4)^2} = \sqrt{2^2 + 0^2} = 2$

$CD = \sqrt{(1-4)^2 + (4-0)^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{9+16} = 5$

$DA = \sqrt{(4-(-4))^2 + (0-0)^2} = \sqrt{8^2 + 0^2} = 8$

Perimeter of trapezium: $P = AB + BC + CD + DA = 5 + 2 + 5 + 8 = 20$.

55. Given $A = (4, -1)$, let $B$ be the reflection of $A$ across $y = x$.

    Reflection across $y = x$ swaps the coordinates: if $A = (x, y)$, then $B = (y, x)$.

    Thus $B = (-1, 4)$

    Distance $AB$: $AB = \sqrt{(-1-4)^2 + (4-(-1))^2} = 5\sqrt{2}$.

56. Given $A = (2, 0)$ and $B = \left(2 + \sqrt{3}, 1\right)$. Rotate $B$ about $A$ by $15^\circ$ anticlockwise.

    Rotation formula about $A$:

    $x' = A_x + (B_x - A_x)\cos(15^\circ) - \left(B_y - A_y\right)\sin(15^\circ), y' = A_y + (B_x - A_x)\sin(15^\circ) + \left(B_y - A_y\right)\cos(15^\circ)$

    $\left(B_x - A_x, B_y - A_y\right) = \left(\sqrt{3}, 1\right)$, $\cos(15^\circ) = \frac{\sqrt{6}+\sqrt{2}}{4}$, $\sin(15^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$

    $x' = 2 + \sqrt{3}\frac{\sqrt{6}+\sqrt{2}}{4} - 1\frac{\sqrt{6}-\sqrt{2}}{4} = \frac{11+\sqrt{2}}{4}, y' = 0 + \sqrt{3}\frac{\sqrt{6}-\sqrt{2}}{4} + 1\frac{\sqrt{6}+\sqrt{2}}{4} = \frac{3+\sqrt{2}}{4}$

    Thus, $C = \left(\frac{11+\sqrt{2}}{4}, \frac{3+\sqrt{2}}{4}\right)$.

57. Given point $P = (1, -2)$. Reflecting $P$ in the $x$-axis: $P' = (1, 2)$ (the $y$-coordinate changes sign)

    Translate $P'$ parallel to the positive $x$-axis by 3 units: $P'' = (1 + 3, 2) = (4, 2)$

58. Given $A = (3, 0)$ and $B = (5, 2)$.

    Segment from $A$ to $B$: $\left(B_x - A_x, B_y - A_y\right) = (5 - 3, 2 - 0) = (2, 2)$

    Rotating this segment about $A$ by $45^\circ$ anticlockwise:

    $x_C = A_x + (B_x - A_x)\cos(45^\circ) - \left(B_y - A_y\right)\sin(45^\circ), y_C = A_y + (B_x - A_x)\sin(45^\circ) + \left(B_y - A_y\right)\cos(45^\circ)$

    Since $\cos(45^\circ) = \sin(45^\circ) = \frac{\sqrt{2}}{2}$, $x_C = 3 + 2\frac{\sqrt{2}}{2} - 2\frac{\sqrt{2}}{2} = 3 + 0 = 3$

    $y_C = 0 + 2\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2} = 0 + 2\frac{\sqrt{2}}{2} + 2\frac{\sqrt{2}}{2} = 2\sqrt{2}$

    So $C = \left(3, 2\sqrt{2}\right)$. Reflecting $C$ in the $y$-axis: $D = (-x_C, y_C) = \left(-3, 2\sqrt{2}\right)$.

59. Let $A(5, -2)$ and $B(9, 6)$.

    For a point dividing the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ in the ratio $m : n$:

    Internal division formula is $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$.

    External division formula is $\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$.

    Here $m = 3$ and $n = 1$.

Internal division point is $\left(\frac{3.9+1.5}{4}, \frac{3.6+1.(-2)}{4}\right) = (8, 4)$.

External division point is $\left(\frac{3.9-1.5}{2}, \frac{3.6-1.(-2)}{2}\right) = (11, 10)$.

60. The roots of $x^2 + 4x + 3 = 0$ are $-3$ and $-1$. Since $x_B < x_C$, we have $x_B = -3$ and $x_C = -1$.

    The roots of $x^2 - x - 6 = 0$ are 3 and $-2$. Since $y_B > y_C$, we have $y_B = 3$ and $y_C = -2$.

    So the coordinates are $B(-3, 3)$, $C(-1, -2)$, and $A(3, -5)$.

    Now, $BC = \sqrt{(-1+3)^2 + (-2-3)^2} = \sqrt{4+25} = \sqrt{29}$,

    $CA = \sqrt{(3+1)^2 + (-5+2)^2} = \sqrt{16+9} = 5$,

    $AB = \texttt{sqrt}\{(3+3)^2 + (-5-3)^2\} = \texttt{sqrt}\{36+64\} = 10$.

    Let $a = BC = \sqrt{29}$, $b = CA = 5$, and $c = AB = 10$.

    The length of the internal angle bisector at $A$ is $l_a = \frac{\sqrt{bc((b+c)^2 - a^2)}}{b+c}$.

    Substituting values, $l_a = \frac{\sqrt{5.10(15^2-29)}}{15} = \frac{\sqrt{50.196}}{15} = \frac{70\sqrt{2}}{15} = \frac{14\sqrt{2}}{3}$.

61. Let the points be $A(4, 3)$ and $B(6, 3)$. Let $P(2, y)$ divide $AB$ in the ratio $m : n$.

    Using the section formula for the x-coordinate, $2 = \frac{6m+4n}{m+n}$.

    So, $2m + 2n = 6m + 4n \Rightarrow 4m + 2n = 0 \Rightarrow m : n = -1 : 2$.

    Hence, the point divides the line segment externally in the ratio $1 : 2$.

    Now using the y-coordinate formula, $y = \frac{3m+3n}{m+n} \Rightarrow y = 3$.

62. The given points are $A(-2, 3)$, $B(8, 9)$, $C(0, 4)$ and $D(3, 0)$.

    Let the lines $AB$ and $CD$ intersect at point $P$.

    Equation of $AB$ in parametric form is $x = -2 + 10t$, $y = 3 + 6t$.

    Equation of $CD$ in parametric form is $x = 3s$, $y = 4 - 4s$.

    At the point of intersection, $-2 + 10t = 3s \Rightarrow 3 + 6t = 4 - 4s$.

    From the first equation, $s = \frac{-2+10t}{3}$.

    Substituting in the second equation, $3 + 6t = 4 - 4\left(\frac{-2+10t}{3}\right) \Rightarrow t = \frac{11}{58}$.

    Therefore, $AP : PB = t : (1-t) = \left(\frac{11}{58}\right) : \left(\frac{47}{58}\right) = 11 : 47$

63. Let $A_1(x_1, y_1), A_2(x_2, y_2), A_3(x_3, y_3), ..., A_{n(x_n, y_n)}$ be $n$ points.

    $A_1A_2$ is bisected at $G_1$, so $G_1 = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

    $G_1A_3$ is divided at $G_2$ in the ratio $1 : 2$, hence $G_2 = \left(\frac{1x_3+2x_{\{G_1\}}}{3}, \frac{1y_3+2y_{\{G_1\}}}{3}\right)$.

    Substituting the coordinates of $G_1$, $G_2 = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

    Now $G_2A_4$ is divided at $G_3$ in the ratio $1 : 3$, so $G_3 = \left(\frac{1x_4+3x_{\{G_2\}}}{4}, \frac{1y_4+3y_{\{G_2\}}}{4}\right)$.

Substituting the coordinates of $G_2$, $G_3 = \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}\right)$.

Proceeding similarly, after dividing $G_{n-2}A_n$ at $G_{n-1}$ in the ratio $1 : (n-1)$, we get $G_{n-1} = \left(\frac{x_1+x_2+\cdots+x_n}{n}, \frac{y_1+y_2+\cdots+y_n}{n}\right)$.

64. Let the straight line be $ax+by+c=0$ and let it intersect the line joining $A(x_1,y_1)$ and $B(x_2,y_2)$ at the point $P$.

Suppose $P$ divides $AB$ in the ratio $m:n$.

Then, by the section formula, the coordinates of $P$ are $x = \frac{mx_2+nx_1}{m+n}$ and $y = \frac{my_2+ny_1}{m+n}$

Since $P$ lies on the line $ax+by+c=0$, we have $\frac{a(mx_2+nx_1)}{m+n} + \frac{b(my_2+ny_1)}{m+n} + c = 0$.

Multiplying by $(m+n)$, $m(ax_2+by_2+c)+n(ax_1+by_1+c)=0$.

Hence, $m(ax_2+by_2+c) = -n(ax_1+by_1+c)$.

Therefore, $m:n = -(ax_1+by_1+c):(ax_2+by_2+c)$.

The negative sign means that the quantities $ax_1+by_1+c$ and $ax_2+by_2+c$ are of opposite signs.

Geometrically, this implies that the points $A(x_1,y_1)$ and $B(x_2,y_2)$ lie on opposite sides of the line $ax+by+c=0$.

65. Let the line $L$ intersect $BC$ at $P$, $CA$ at $Q$, and $AB$ at $R$.

Equation of the line $L$ may be written as $ax+by+c=0$.

Since $P$ lies on $BC$, by the result for division of a line by a straight line, $\frac{BP}{PC} = -\frac{ax_2+by_2+c}{ax_3+by_3+c}$.

Similarly, since $Q$ lies on $CA$, $\frac{CQ}{QA} = -\frac{ax_3+by_3+c}{ax_1+by_1+c}$.

And since $R$ lies on $AB$, $\frac{AR}{RB} = -\frac{ax_1+by_1+c}{ax_2+by_2+c}$.

Multiplying the three ratios, we get $\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB} = (-1)^3\frac{(ax_2+by_2+c)(ax_3+by_3+c)(ax_1+by_1+c)}{(ax_3+by_3+c)(ax_1+by_1+c)(ax_2+by_2+c)} = -1$.

66. The vertices of the triangle are $A = (x_1, x_1\tan\alpha)$, $B = (x_2, x_2\tan\beta)$, $C = (x_3, x_3\tan\gamma)$.

Since the circumcenter is at the origin, we have $x_1^2 + (x_1\tan\alpha)^2 = x_2^2 + (x_2\tan\beta)^2 = x_3^2 + (x_3\tan\gamma)^2$.

Hence $x_1\sec\alpha = x_2\sec\beta = x_3\sec\gamma = R$.

The coordinates of the orthocenter $H(\overline{x},\overline{y})$ are $\overline{x} = x_1+x_2+x_3$, $\overline{y} = x_1\tan\alpha + x_2\tan\beta + x_3\tan\gamma$.

Substituting $x_i = R\cos\theta_i$, we get $\overline{x} = R(\cos\alpha+\cos\beta+\cos\gamma)$, $\overline{y} = R(\cos\alpha\tan\alpha+\cos\beta\tan\beta+\cos\gamma\tan\gamma) = R(\sin\alpha+\sin\beta+\sin\gamma)$.

Therefore, $\frac{\overline{y}}{\overline{x}} = \frac{\sin\alpha + \sin\beta + \sin\gamma}{\cos\alpha + \cos\beta + \cos\gamma}$.

67. The vertices of the triangle are $A\left(\alpha, \frac{1}{\alpha}\right)$, $B\left(\beta, \frac{1}{\beta}\right)$, and $C\left(\gamma, \frac{1}{\gamma}\right)$, where $\alpha$, $\beta$, and $\gamma$ are the roots of $x^3 - 3px^2 + 3qx - 1 = 0$.

    The centroid of the triangle is $G = \left(\frac{\alpha+\beta+\gamma}{3}, \frac{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}{3}\right)$.

    Since $\alpha + \beta + \gamma = 3p$ and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma} = 3q$, we have

    $G = \left(\frac{3p}{3}, \frac{3q}{3}\right) = (p, q)$.

68. Let the points be $A(at^2, 2at)$, $B\left(\frac{a}{t^2}, -2\frac{a}{t}\right)$, and $C(a, 0)$.

    The distance between $A$ and $C$ is $AC = \sqrt{\left(at^2 - a\right)^2 + (2at - 0)^2} = a(t^2 + 1)$

    The distance between $B$ and $C$ is $BC = \sqrt{\left(\frac{a}{t^2} - a\right)^2 + \left(-2\frac{a}{t} - 0\right)^2} = a\frac{t^2+1}{t^2}$

    Then $\frac{1}{AC} + \frac{1}{BC} = \frac{1}{a(t^2+1)} + \frac{t^2}{a(t^2+1)} = \frac{1}{a}$

    Hence, $\frac{1}{AC} + \frac{1}{BC}$ is independent of $t$.

69. Let $A = (0, 0)$, $B = \left(3, \sqrt{3}\right)$, and $C = (x, y)$. $AB = 2\sqrt{3}$, so $AC = BC = 2\sqrt{3}$.

    From distances: $x^2 + y^2 = 12$, $(x - 3)^2 + \left(y - \sqrt{3}\right)^2 = 12$.

    Simplifying gives $y = 2\sqrt{3} - \sqrt{3}x$.

    Substituting in $x^2 + y^2 = 12$: $x^2 + \left(2\sqrt{3} - \sqrt{3}x\right)^2 = 12 \Rightarrow 4x(x - 3) = 0 \Rightarrow x = 0 \quad \text{or} \quad 3$.

    Then $y = 2\sqrt{3}$ or $y = -\sqrt{3}$.

    Hence $C = \left(0, 2\sqrt{3}\right)$ or $C = \left(3, -\sqrt{3}\right)$.

70. Let the vertices of the triangle be $A(-2, 3)$, $B(2, -1)$, $C(4, 0)$.

    The circumcenter $(x, y)$ is given by

    $x = \frac{(x_1^2+y_1^2)(y_2-y_3)+(x_2^2+y_2^2)(y_3-y_1)+(x_3^2+y_3^2)(y_1-y_2)}{2(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2))}$

    $y = \frac{(x_1^2+y_1^2)(x_3-x_2)+(x_2^2+y_2^2)(x_1-x_3)+(x_3^2+y_3^2)(x_2-x_1)}{2(x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2))}$

    Denominator: $D = 2((-2)(-1 - 0) + 2(0 - 3) + 4(3 - (-1))) = 24$

    $x = \frac{(-2^2+3^2)(-1-0)+(2^2+(-1)^2)(0-3)+(4^2+0^2)(3-(-1))}{24} = 1.5$

    $y = \frac{(-2^2+3^2)(4-2)+(2^2+(-1)^2)(-2-4)+(4^2+0^2)(2-(-2))}{24} = 2.5$

    Hence the circumcenter is $O(1.5, 2.5)$.

    The circumradius is the distance from $O$ to any vertex, say $A$:

    $R = \sqrt{(1.5 - (-2))^2 + (2.5 - 3)^2} = \sqrt{3.5^2 + (-0.5)^2} = \sqrt{12.25 + 0.25} = \sqrt{12.5} = 5\frac{\sqrt{2}}{2}$

71. Given $A = (1, 1)$, $B = (4, 5)$, and $C = (6, 13)$.

$AB = \sqrt{(4-1)^2 + (5-1)^2} = \sqrt{9+16} = 5$

$AC = \sqrt{(6-1)^2 + (13-1)^2} = \sqrt{25+144} = 13$

$BC = \sqrt{(6-4)^2 + (13-5)^2} = \sqrt{4+64} = \sqrt{68}$

Using cosine law at angle $A$:

$\cos A = \frac{AB^2 + AC^2 - BC^2}{2ABAC}$

$\cos A = \frac{25+169-68}{2.5.13} = \frac{63}{65}$.

72. The given points are $\left(3, \frac{\pi}{4}\right)$ and $\left(7, 5\frac{\pi}{4}\right)$.

    Distance between two points in polar coordinates is: $d^2 = r_1^2 + r_2^2 - 2.r_1.r_2.\cos(\theta_2 - \theta_1)$

    Here, $r_1 = 3$, $r_2 = 7$ and $\theta_2 - \theta_1 = 5.\frac{\pi}{4} - \frac{\pi}{4} = \pi$. Now $\cos(\pi) = -1$

    Substituting: $d^2 = 3^2 + 7^2 - 2.3.7.(-1) \quad \Rightarrow d^2 = 9 + 49 + 42 = 100 \quad \Rightarrow d = \sqrt{100} = 10$.

73. Given $A = (2,4)$ and $B = (2,6)$. Length $AB = \sqrt{(2-2)^2 + (6-4)^2} = 2$.

    Midpoint of $AB$: $M = \left(\frac{2+2}{2}, \frac{4+6}{2}\right) = (2,5)$

    Since $AB$ is vertical, the equilateral triangle lies horizontally. Height of an equilateral triangle with side 2 is: $h = \sqrt{3}$

    The origin $(0,0)$ lies to the left of line $x = 2$, so the vertex opposite to the origin lies to the right.

    Hence, $P = \left(2 + \sqrt{3}, 5\right)$

74. Given points in polar coordinates: $A = (2, 45^\circ)$, $B = \left(\sqrt{2}, 90^\circ\right)$, and $C = (-2, 135^\circ)$.

    Converting to cartesian coordinates using $x = r\cos(\theta)$ and $y = r\sin(\theta)$

    $A = (2\cos(45^\circ), 2\sin(45^\circ)) = \left(\sqrt{2}, \sqrt{2}\right)$, $\quad B = \left(\sqrt{2}\cos(90^\circ), \sqrt{2}\sin(90^\circ)\right) = \left(0, \sqrt{2}\right)$, and $C = (-2\cos(135^\circ), -2\sin(135^\circ)) = \left(\sqrt{2}, -\sqrt{2}\right)$

    Finding squares of side lengths using distance formula: $AB^2 = \left(0 - \sqrt{2}\right)^2 + \left(\sqrt{2} - \sqrt{2}\right)^2 = 2$

    $AC^2 = \left(\sqrt{2} - \sqrt{2}\right)^2 + \left(-\sqrt{2} - \sqrt{2}\right)^2 = 8$

    $BC^2 = \left(\sqrt{2} - 0\right)^2 + \left(-\sqrt{2} - \sqrt{2}\right)^2 = 10$

    $AB^2 + AC^2 = 2 + 8 = 10 = BC^2$. And hence, the triangle is a right-angled triangle.

75. Let $A = (2,4)$ and $B = (6,8)$.

    Internal division in the ratio $1:3$: $x = \frac{1.6+3.2}{1+3} = \frac{12}{4} = 3$, and $y = \frac{1.8+3.4}{1+3} = \frac{20}{4} = 5$

    So, the point of internal division is $(3,5)$.

External division in the ratio $1:3$: $x = \frac{1.6-3.2}{1-3} = \frac{0}{-2} = 0$, and $y = \frac{1.8-3.4}{1-3} = \frac{-4}{-2} = 2$

So, the point of external division is $(0, 2)$.

76. The solutions are given below:

    1. Let $A = (1, -2)$ and $B = (-3, 4)$. The trisection points divide the line internally in the ratios $1:2$ and $2:1$.

       First point (ratio $1:2$): $x = \frac{1.(-3)+2.1}{1+2} = \frac{-3+2}{3} = -\frac{1}{3}$, and $y = \frac{1.4+2.(-2)}{1+2} = \frac{4-4}{3} = 0$

       So, the first trisection point is $\left(-\frac{1}{3}, 0\right)$.

       Second point (ratio $2:1$): $x = \frac{2.(-3)+1.1}{2+1} = \frac{-6+1}{3} = -\frac{5}{3}$, and $y = \frac{2.4+1.(-2)}{2+1} = \frac{8-2}{3} = 2$

       So, the second trisection point is $\left(-\frac{5}{3}, 2\right)$.

    2. Let $A = (2, 3)$ and $B = (6, 5)$. The trisection points divide the line internally in the ratios $1:2$ and $2:1$.

       First point (ratio $1:2$): $x = \frac{1.6+2.2}{1+2} = \frac{6+4}{3} = \frac{10}{3}$, and $y = \frac{1.5+2.3}{1+2} = \frac{5+6}{3} = \frac{11}{3}$

       So, the first trisection point is $\left(\frac{10}{3}, \frac{11}{3}\right)$.

       Second point (ratio $2:1$): $x = \frac{2.6+1.2}{2+1} = \frac{12+2}{3} = \frac{14}{3}$, and $y = \frac{2.5+1.3}{2+1} = \frac{10+3}{3} = \frac{13}{3}$

       So, the second trisection point is $\left(\frac{14}{3}, \frac{13}{3}\right)$.

77. Let $A = (1, 1)$ and $B = (2, -3)$. Point $D$ lies on $AB$ produced such that $AD = 3AB$.

    The ratio of division for external point $D$ is $A : B = 3 : -1$.

    Coordinates of $D$ using external division: $x_D = \frac{3.2-1.1}{3-1} = \frac{6-1}{2} = \frac{5}{2}$, and $y_D = \frac{3.(-3)-1.1}{3-1} = \frac{-9-1}{2} = -\frac{10}{2} = -5$

    So, the coordinates of $D$ are $\left(\frac{5}{2}, -5\right)$.

78. Let the points be $A = (3, 4)$ and $B = (k, 7)$.

    Midpoint $M = (x, y)$ is given by: $x = \frac{3+k}{2}$, and $y = \frac{4+7}{2} = \frac{11}{2}$

    The line passing through $M$ satisfies: $2x + 2y + 1 = 0$

    Substituting $x$ and $y$: $2.\left(\frac{3+k}{2}\right) + 2.\left(\frac{11}{2}\right) + 1 = 0$

    $\Rightarrow (3 + k) + 11 + 1 = 0 \Rightarrow k + 15 = 0 \Rightarrow k = -15$.

79. Let one end of the diameter be $A = (2, 3)$ and the center be $O = (-2, 5)$. Let the other end be $B = (x, y)$.

    The midpoint of $AB$ is the center $O$, so $O_x = \frac{A_x+B_x}{2}$, $O_y = \frac{A_y+B_y}{2}$

    Substituting the known values: $-2 = \frac{2+x}{2} \Rightarrow 2 + x = -4 \Rightarrow x = -6$

$5 = \frac{3+y}{2} \Rightarrow 3 + y = 10 \Rightarrow y = 7$

Thus, the coordinates of the other end of the diameter are: $B = (-6, 7)$.

80. Let the vertices be $A = (-1, 3)$, $B = (1, -1)$, $C = (5, 1)$.

    Median from $A$ to midpoint of $BC$: Midpoint of $BC$: $M_a = \left(\frac{1+5}{2}, \frac{-1+1}{2}\right) = \left(\frac{6}{2}, \frac{0}{2}\right) = (3, 0)$

    Length of median $AM_a$: $AM_a = \sqrt{(3-(-1))^2 + (0-3)^2} = \sqrt{4^2 + (-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

    Median from $B$ to midpoint of $AC$: Midpoint of $AC$: $M_b = \left(\frac{-1+5}{2}, \frac{3+1}{2}\right) = \left(\frac{4}{2}, \frac{4}{2}\right) = (2, 2)$

    Length of median $BM_b$: $BM_b = \sqrt{(2-1)^2 + (2-(-1))^2} = \sqrt{1^2 + 3^2} = \sqrt{1+9} = \sqrt{10}$

    Median from $C$ to midpoint of $AB$: Midpoint of $AB$: $M_c = \left(\frac{-1+1}{2}, \frac{3+(-1)}{2}\right) = \left(\frac{0}{2}, \frac{2}{2}\right) = (0, 1)$

    Length of median $CM_c$: $CM_c = \sqrt{(0-5)^2 + (1-1)^2} = \sqrt{(-5)^2 + 0^2} = \sqrt{25} = 5$.

81. Let $A = (2, 5)$, $C = (-1, 2)$, and $B = (x, y)$. Point $C$ divides $AB$ in the ratio $3 : 4$.

    Using section formula for internal division:

    $C_x = \frac{4.2+3.x}{3+4} \Rightarrow -1 = \frac{8+3.x}{7} \Rightarrow 8 + 3.x = -7 \Rightarrow 3.x = -15 \Rightarrow x = -5$

    $C_y = \frac{4.5+3.y}{3+4} \Rightarrow 2 = \frac{20+3.y}{7} \Rightarrow 20 + 3.y = 14 \Rightarrow 3.y = -6 \Rightarrow y = -2$

82. Let $A = (3, 4)$, $B = (7, 7)$, $C = (x, y)$ with $B$ between $A$ and $C$.

    Distance $AB = \sqrt{(7-3)^2 + (7-4)^2} = 5$. Distance $AC = 10 \Rightarrow BC = AC - AB = 5$

    Since $B$ divides $AC$ internally in the ratio $AB : BC = 5 : 5 = 1 : 1$,

    Use section formula for internal division: $B_x = \frac{x+3}{2} \Rightarrow 7 = \frac{x+3}{2} \Rightarrow x = 11$, and $B_y = \frac{y+4}{2} \Rightarrow 7 = \frac{y+4}{2} \Rightarrow y = 10$.

83. Let $A = (2, -2)$, $B = (-4, 1)$, and $P = (-8, 3)$ divides $AB$ in the ratio $k : 1$.

    Using section formula: $P_x = \frac{k.(-4)+1.2}{k+1} \Rightarrow -8 = \frac{-4k+2}{k+1}$

    Multiply both sides by $(k+1)$: $-8.(k+1) = -4k + 2 \Rightarrow -8k - 8 = -4k + 2 \Rightarrow -8k + 4k = 2 + 8 \Rightarrow -4k = 10 \Rightarrow k = -\frac{5}{2}$

84. Let $A = (2, -3)$, $B = (5, 6)$, and $P = (x, 0)$ be the point where the line meets the $x$-axis.

    Suppose $P$ divides $AB$ in the ratio $k : 1$. Using section formula for $y$-coordinate:

    $0 = \frac{k.6+1.(-3)}{k+1} \Rightarrow 0 = \frac{6k-3}{k+1} \Rightarrow 6k - 3 = 0 \Rightarrow 6k = 3 \Rightarrow k = \frac{1}{2}$

85. Let $C = (-1, 2)$, $D = (4, -5)$, and $P$ be the intersection of line $AB$ with $CD$. Suppose $P$ divides $CD$ in the ratio $m : n$.

    $P_x = \frac{-n+4m}{m+n}$, $P_y = \frac{2n-5m}{m+n}$

    Line $AB$: $X = 15t$, $Y = -1 + 3t$

    Equating coordinates: $15t = \frac{-n+4m}{m+n}$, and $-1 + 3t = \frac{2n-5m}{m+n}$

    Eliminate $t$: $\frac{-n+4m}{15} = \frac{-4m+3n}{3} \Rightarrow 24m = 16n \Rightarrow m : n = 2 : 3$

86. Let $A = (1, 2)$, $B = (-2, 3)$, and $P = (x, y)$ be the point where $AB$ meets the line $3x + 4y = 7$.

    Suppose $P$ divides $AB$ in the ratio $k : 1$.

    Coordinates of $P$ using section formula: $x = \frac{k.(-2)+1.1}{k+1} = \frac{-2k+1}{k+1}$, and $y = \frac{k.3+1.2}{k+1} = \frac{3k+2}{k+1}$

    Substituting into the line equation $3x + 4y = 7$: $3.\left(\frac{-2k+1}{k+1}\right) + 4.\left(\frac{3k+2}{k+1}\right) = 7$

    $\Rightarrow \frac{-6k+3+12k+8}{k+1} = 7 \Rightarrow \frac{6k+11}{k+1} = 7$

    $\Rightarrow 6k + 11 = 7.(k + 1) \Rightarrow 6k + 11 = 7k + 7 \Rightarrow k = 4.$

87. Let $A = (3, -1)$, $B = (8, 9)$, and $P = (x, y)$ be the point where the line $y - x + 2 = 0$ meets $AB$. Suppose $P$ divides $AB$ in the ratio $k : 1$.

    Coordinates of $P$ using section formula: $x = \frac{k.8+1.3}{k+1} = \frac{8k+3}{k+1}$, and $y = \frac{k.9+1.(-1)}{k+1} = \frac{9k-1}{k+1}$

    Substituting into the line equation $y - x + 2 = 0$: $\frac{9k-1}{k+1} - \frac{8k+3}{k+1} + 2 = 0$

    Combining terms: $\frac{9k-1-8k-3}{k+1} + 2 = 0 \Rightarrow \frac{k-4}{k+1} + 2 = 0$

    $\frac{k-4+2.(k+1)}{k+1} = 0 \Rightarrow \frac{k-4+2k+2}{k+1} = 0 \Rightarrow \frac{3k-2}{k+1} = 0 \Rightarrow 3k - 2 = 0 \Rightarrow k = \frac{2}{3}.$

88. Let $A = (5, -4)$, $B = (3, -2)$, and $P$ divides $AB$ in the ratio $4 : 3$.

    Coordinates of $P$ using section formula: $x = \frac{3.5+4.3}{4+3} = \frac{15+12}{7} = \frac{27}{7}$, and $y = \frac{3.(-4)+4.(-2)}{4+3} = \frac{-12-8}{7} = -\frac{20}{7}$

    Distance of $P$ from origin: $OP = \sqrt{\left(\frac{27}{7}\right)^2 + \left(-\frac{20}{7}\right)^2} = \sqrt{\frac{729}{49} + \frac{400}{49}} = \sqrt{\frac{1129}{49}} = \frac{\sqrt{1129}}{7}.$

89. Let the vertices be $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.

    Midpoints of sides: $M_{AB} = (1, 1)$, $M_{BC} = (2, 3)$, $M_{CA} = (4, 1)$

    Midpoint formula: $M_{AB} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (1, 1) \Rightarrow x_1 + x_2 = 2, y_1 + y_2 = 2$

    $M_{BC} = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right) = (2, 3) \Rightarrow x_2 + x_3 = 4, y_2 + y_3 = 6$

    $M_{CA} = \left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right) = (4, 1) \Rightarrow x_3 + x_1 = 8, y_3 + y_1 = 2$

    Solving for $x$-coordinates: $x_1 + x_2 = 2$, $x_2 + x_3 = 4$, $x_3 + x_1 = 8$

Adding first two: $x_1 + 2x_2 + x_3 = 6$

Subtracting third: $(x_1 + 2x_2 + x_3) - (x_3 + x_1) = 6 - 8 \Rightarrow 2x_2 - x_3 = -2 \Rightarrow x_3 = 2x_2 + 2$

From $x_2 + x_3 = 4 \Rightarrow x_2 + 2x_2 + 2 = 4 \Rightarrow 3x_2 = 2 \Rightarrow x_2 = \frac{2}{3}$ Then $x_3 = 2.\left(\frac{2}{3}\right) + 2 = \frac{10}{3}$, $x_1 = 2 - x_2 = \frac{4}{3}$

Solving for $y$-coordinates: $y_1 + y_2 = 2$, $y_2 + y_3 = 6$, $y_3 + y_1 = 2$

Adding first two: $y_1 + 2y_2 + y_3 = 8$

Subtracting third: $(y_1 + 2y_2 + y_3) - (y_3 + y_1) = 8 - 2 \Rightarrow 2y_2 = 6 \Rightarrow y_2 = 3$

Then $y_1 = 2 - 3 = -1$, $y_3 = 2 - y_1 = 3$.

90. The solutions are given below:

1. Let $A = (2, 4)$, $B = (6, 4)$, $C = (2, 0)$.

   Centroid $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

   $G = \left(\frac{2+6+2}{3}, \frac{4+4+0}{3}\right) = \left(\frac{10}{3}, \frac{8}{3}\right)$

   Side lengths:

   $AB = \sqrt{(6-2)^2 + (4-4)^2} = \sqrt{16} = 4$

   $BC = \sqrt{(6-2)^2 + (4-0)^2} = \sqrt{16+16} = \sqrt{32} = 4.\sqrt{2}$

   $CA = \sqrt{(2-2)^2 + (0-4)^2} = \sqrt{16} = 4$

   Incenter formula: $I = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

   Here, $a = BC = 4.\sqrt{2}$, $b = CA = 4$, $c = AB = 4$

   $x$-coordinate: $I_x = \frac{a.2+b.6+c.2}{a+b+c} = \frac{4.\sqrt{2}.2+4.6+4.2}{4.\sqrt{2}+4+4}$

   $I_x = \frac{8.\sqrt{2}+24}{4.\sqrt{2}+8}$

   $y$-coordinate: $I_y = \frac{a.4+b.4+c.0}{a+b+c} = \frac{16.\sqrt{2}+16}{4.\sqrt{2}+8}$.

2. Let $A = (1, 2)$, $B = (2, 3)$, $C = (3, 4)$.

   Centroid $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

   $G = \left(\frac{1+2+3}{3}, \frac{2+3+4}{3}\right) = \left(\frac{6}{3}, \frac{9}{3}\right) = (2, 3)$

   Side lengths: $AB = \sqrt{(2-1)^2 + (3-2)^2} = \sqrt{1+1} = \sqrt{2}$

   $BC = \sqrt{(3-2)^2 + (4-3)^2} = \sqrt{1+1} = \sqrt{2}$

   $CA = \sqrt{(3-1)^2 + (4-2)^2} = \sqrt{4+4} = \sqrt{8} = 2.\sqrt{2}$

   Incenter formula: $I = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

   Let $a = BC = \sqrt{2}$, $b = CA = 2.\sqrt{2}$, $c = AB = \sqrt{2}$

   $x$-coordinate: $I_x = \frac{a.1+b.2+c.3}{a+b+c} = \frac{\sqrt{2}.1+2.\sqrt{2}.2+\sqrt{2}.3}{\sqrt{2}+2.\sqrt{2}+\sqrt{2}}$

$I_x = \frac{1.\sqrt{2}+4.\sqrt{2}+3.\sqrt{2}}{4.\sqrt{2}} = 8.\frac{\sqrt{2}}{4.\sqrt{2}} = 2$

$y$-coordinate: $I_y = \frac{a.2+b.3+c.4}{a+b+c} = \frac{2.\sqrt{2}+6.\sqrt{2}+4.\sqrt{2}}{4.\sqrt{2}} = 12.\frac{\sqrt{2}}{4.\sqrt{2}} = 3$

91. Let $A = (-1, 4)$, $B = (5, 2)$, $C = (x, y)$, and centroid $G = (0, -3)$.

Centroid formula: $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

$x$-coordinate: $0 = \frac{-1+5+x}{3} \Rightarrow 4 + x = 0 \Rightarrow x = -4$

$y$-coordinate: $-3 = \frac{4+2+y}{3} \Rightarrow 6 + y = -9 \Rightarrow y = -15$

92. Let $A = (1, 4)$, $B = (4, 8)$, and $P$ divides $AB$ externally in the ratio $m : 1$.

Let $AP = 10$ and $AB = \sqrt{(4-1)^2 + (8-4)^2} = \sqrt{9+16} = 5$

Since $AP = AB + BP$, $P$ divides $AB$ externally => ratio $m : 1 = 2 : 1$

Coordinates of $P$ using external division formula: $x = \frac{m.4-1}{m-1} = \frac{2.4-1}{2-1} = \frac{8-1}{1} = 7$, and $y = \frac{m.8-4}{m-1} = \frac{2.8-4}{2-1} = \frac{16-4}{1} = 12$.

93. Let $A = (3, 4), B = (-4, 3), C = (8, 6)$.

Area formula: $\Delta = \left(\frac{1}{2}\right) |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

$= \left(\frac{1}{2}\right) |3.(3-6) + (-4).(6-4) + 8.(4-3)| = \frac{9}{2}$.

94. Let the vertices be $A(-3, 2), B(7, -6), C(-5, -4), D(5, 4)$, taken in order.

Area formula for quadrilateral: $\Delta = \frac{1}{2} |x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - (y_1x_2 + y_2x_3 + y_3x_4 + y_4x_1)|$

Substituting values:

$S_1 = (-3).(-6) + 7.(-4) + (-5).4 + 5.2 = 18 - 28 - 20 + 10 = -20$

$S_2 = 2.7 + (-6).(-5) + (-4).5 + 4.(-3) = 14 + 30 - 20 - 12 = 12$

$\Delta = \left(\frac{1}{2}\right) |-20 - 12| = \left(\frac{1}{2}\right).32 = 16$.

95. Let A = (6, 3), B = (-3, 5), C = (4, -2), P = (x, y).

Area of a triangle with vertices (X_1, Y_1), (X_2, Y_2), (X_3, Y_3): $\Delta = \frac{1}{2} |X_1.(Y_2 - Y_3) + X_2.(Y_3 - Y_1) + X_3.(Y_1 - Y_2)|$

$\Delta ABC = \left(\frac{1}{2}\right) |6.(5-(-2)) + (-3).(-2-3) + 4.(3-5)| = \left(\frac{1}{2}\right).49 = \frac{49}{2}$

$\Delta PBC = \left(\frac{1}{2}\right) |x.(5-(-2)) + (-3).(-2-y) + 4.(y-5)| = \left(\frac{7}{2}\right) |x+y-2|$

Now, ratio: $\frac{\Delta PBC}{\Delta ABC} = \frac{\frac{7}{2}|x+y-2|}{\frac{49}{2}} = \frac{|x+y-2|}{7}$.

96. Let $A = (3, 3)$, $B = (h, 0)$, $C = (0, k)$.

Points are collinear if the area of triangle $ABC$ is 0: $\Delta ABC = \left(\frac{1}{2}\right) |x_1.(y_2 - y_3) + x_2.(y_3 - y_1) + x_3.(y_1 - y_2)|$

Substituting values: $\Delta ABC = \left(\frac{1}{2}\right) |3.(0-k) + h.(k-3) + 0.(3-0)| = \left(\frac{1}{2}\right) |h.k - 3.h - 3.k|$

For collinearity, $\Delta ABC = 0 \Rightarrow h.k - 3.h - 3.k = 0$

Divide by $h.k$ ($h \neq 0,\ k \neq 0$): $1 - \frac{3}{k} - \frac{3}{h} = 0 \Rightarrow \frac{1}{h} + \frac{1}{k} = \frac{1}{3}$.

97. Let $A = (x_1, y_1),\ B = (x_2, y_2),\ C = (x_3, y_3)$. Let $(x, y)$ be a point on the internal bisector of $\angle A$. Let $AB = c,\ AC = b$.

The area of a triangle can be expressed using the determinant: $\Delta(X, Y, Z) = \begin{vmatrix} X_x & X_y & 1 \\ Y_x & Y_y & 1 \\ Z_x & Z_y & 1 \end{vmatrix}$

Let $\Delta PAB = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$ Let $\Delta PAC = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

By the angle bisector theorem, a point on the bisector divides the opposite side in the ratio of adjacent sides: $\frac{BP}{PC} = \frac{AB}{AC} = \frac{c}{b}$

The signed areas satisfy the same ratio: $\frac{\Delta PAB}{\Delta PAC} = -\frac{PC}{PB} = -\frac{b}{c}$

Cross multiplying: $b.\Delta PAB + c.\Delta PAC = 0$.

98. Let the points be: $P(a) = \left(\frac{a^3}{a-1}, \frac{a^2-3}{a-1}\right),\ Q(b) = \left(\frac{b^3}{b-1}, \frac{b^2-3}{b-1}\right),\ R(c) = \left(\frac{c^3}{c-1}, \frac{c^2-3}{c-1}\right)$.

Three points are collinear if the determinant vanishes: $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Substitute $x_i$ and $y_i$: $\begin{vmatrix} \frac{a^3}{a-1} & \frac{a^2-3}{a-1} & 1 \\ \frac{b^3}{b-1} & \frac{b^2-3}{b-1} & 1 \\ \frac{c^3}{c-1} & \frac{c^2-3}{c-1} & 1 \end{vmatrix} = 0$

Multiplying each row by its denominator to simplify:

$\begin{vmatrix} a^3 & a^2-3 & a-1 \\ b^3 & b^2-3 & b-1 \\ c^3 & c^2-3 & c-1 \end{vmatrix} = 0$

Expanding the determinant and simplifying (using factorization of cubic polynomials) gives: $abc - (ab + bc + ca) + 3(a + b + c) = 0$.

99. Let the vertices of the triangle be $A = (4, -8),\ B = (-9, 7),\ C = (x, y)$. The centroid $G = (1, 4)$.

Coordinates of the centroid: $x_G = \frac{x_1+x_2+x_3}{3},\ y_G = \frac{y_1+y_2+y_3}{3}$

Substitute values: $1 = \frac{4-9+x}{3} \Rightarrow 1 = \frac{-5+x}{3} \Rightarrow x = 8 \Rightarrow 4 = \frac{-8+7+y}{3} \Rightarrow 4 = \frac{-1+y}{3} \Rightarrow y = 13$

So $C = (8, 13)$.

Area of triangle $ABC$: $\Delta = \frac{1}{2}\ |x_1.(y_2 - y_3) + x_2.(y_3 - y_1) + x_3.(y_1 - y_2)|$

Substituting values: $\Delta = \frac{1}{2}\ |4.(7 - 13) + (-9).(13 + 8) + 8.(-8 - 7)| = 166.5$.

100. Assume an equilateral triangle has vertices $A = (x_1, y_1),\ B = (x_2, y_2),\ C = (x_3, y_3)$ with all coordinates rational.

Distance formula: $AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$,

$BC^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2,$

$CA^2 = (x_1 - x_3)^2 + (y_1 - y_3)^2$

For an equilateral triangle: $AB^2 = BC^2 = CA^2 = d^2$

Consider the line $AB$. The perpendicular from $C$ to $AB$ must satisfy the formula for height of an equilateral triangle: $h = \frac{\text{sqrt\{3\}}}{2}.d$

Coordinates of $C$ satisfy the perpendicular distance formula from line $AB$: $(| \ (y_2 - y_1)x_C - (x_2 - x_1)y_C + (x_2 y_1 - x_1 y_2)) \frac{|}{\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}} = h = \left(\frac{\sqrt{3}}{2}\right)d$

All numbers on the left are rational, but the right-hand side involves sqrt{3}, which is irrational.

This is a contradiction.

Hence, the coordinates of the vertices of an equilateral triangle cannot all be rational

101. Let $A = (-1, 5)$, $B = (3, 1)$, $C = (5, 7)$.

Midpoints: $D = \left(\frac{3+5}{2}, \frac{1+7}{2}\right) = (4, 4)$, $E = \left(\frac{5-1}{2}, \frac{7+5}{2}\right) = (2, 6)$, and $F = \left(\frac{-1+3}{2}, \frac{5+1}{2}\right) = (1, 3)$

$\Delta ABC = \left(\frac{1}{2}\right)|(-1).(1-7) + 3.(7-5) + 5.(5-1)| = 16$

$\Delta DEF = \left(\frac{1}{2}\right)|4.(6-3) + 2.(3-4) + 1.(4-6)| = 4$

Hence, $\Delta ABC = 4\Delta DEF$.

102. Let $A = (3, 0)$, $B = (0, 6)$, $C = (6, 9)$.

Points dividing sides: $D =$ divides AB in 1:2 $= \left(\frac{2.3+1.0}{3}, \frac{2.0+1.6}{3}\right) = (2, 2)$ and $E =$ divides AC in 1:2 $= \left(\frac{2.3+1.6}{3}, \frac{2.0+1.9}{3}\right) = (4, 3)$

$\Delta = \frac{1}{2}|x_1.(y_2 - y_3) + x_2.(y_3 - y_1) + x_3.(y_1 - y_2)|$

$\Delta ABC = \frac{1}{2}|3.(6-9) + 0.(9-0) + 6.(0-6)| = 22.5$

$\Delta ADE = \frac{1}{2}|3.(2-3) + 2.(3-0) + 4.(0-2)| = 2.5$

$\frac{\Delta ABC}{\Delta ADE} = \frac{22.5}{2.5} = 9.$

103. Let the vertices be: $A = (t, t-2)$, $B = (t+3, t)$, $C = (t+2, t+2)$

$\Delta = \frac{1}{2} \ |x_1.(y_2 - y_3) + x_2.(y_3 - y_1) + x_3.(y_1 - y_2)|$

Substitute coordinates: $\Delta = \frac{1}{2} \ | \ t.(t - (t+2)) + (t+3).((t+2) - (t-2)) + (t+2).((t-2) - t) \ |$

$t.(t - t - 2) = t.(-2) = -2t \quad (t+3).(t+2-t+2) = (t+3).4 = 4t + 12 \quad (t+2).(t-2-t) = (t+2).(-2) = -2t - 4$

Adding terms: $-2t + 4t + 12 - 2t - 4 = 8$

Divide by 2: $\Delta = \frac{1}{2}.8 = 4$, which is independent of $t$.

104. Let the vertices be $A = (x, y)$, $B = (1, 2)$, $C = (2, 1)$.

Area formula: $\Delta = \frac{1}{2} \mid x.(y_2 - y_3) + x_2.(y_3 - y) + x_3.(y - y_2) \mid$

Substitute coordinates: $6 = \left(\frac{1}{2}\right) \mid x.(2-1) + 1.(1-y) + 2.(y-2) \mid$

Simplify: $6 = \left(\frac{1}{2}\right) \mid x.1 + 1.(1-y) + 2.(y-2) \mid \Rightarrow 6 = \left(\frac{1}{2}\right) \mid x + y - 3 \mid$

Multiply both sides by 2: $12 = |x + y - 3|$

Solving for absolute value: $x + y - 3 = 12 \Rightarrow x + y = 15$ or $x + y - 3 = -12 \Rightarrow x + y = -9$.

105. Vertices: $A = (1,1)$, $B = (7,3)$, $C = (12,2)$, $D = (7,21)$.

Area: $\Delta = \frac{1}{2}\ |1.(3-21) + 7.(2-1) + 12.(21-3) + 7.(1-2)|\ = \frac{1}{2}\ |-18 + 7 + 216 - 7| = \frac{1}{2}.198 = 99$.

106. Vertices: $A = (4,3)$, $B = (-5,6)$, $C = (0,-7)$, $D = (3,-6)$, $E = (-7,-2)$.

Area: $\Delta = \frac{1}{2}|4.6 + (-5).(-7) + 0.(-6) + 3.(-2) + (-7).3 - (3.(-5) + 6.0 + (-7).3 + (-6).(-7) + (-2).4)|\ = \frac{1}{2}|32 - (-2)| = \frac{1}{2}.34 = 17$

107. Vertices: $A = (5,0)$, $B = (4,2)$, $C = (1,3)$, $D = (-2,2)$, $E = (-3,-1)$, $F = (0,-4)$.

Area: $\Delta = \frac{1}{2}|5.2 + 4.3 + 1.2 + (-2).(-1) + (-3).(-4) + 0.0 - (0.4 + 2.1 + 3.(-2) + 2.(-3) + (-1).0 + (-4).5)|\ = \frac{1}{2}|68| = 34$.

108. Vertices: $A = ((a+1)(a+2), a+2)$, $B = ((a+2)(a+3), a+3)$, $C = ((a+3)(a+4), a+4)$.

Area: $\Delta = \left(\frac{1}{2}\right)|\ (a+1)(a+2).((a+3)-(a+4)) + (a+2)(a+3).((a+4)-(a+2)) + (a+3)(a+4).((a+2)-(a+3))\ | = 0$

Thus, the points are collinear.

109. A divides P(-5,1) and Q(3,5) in ratio k:1: $A = \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$

Area of $\triangle\ ABC = 2$: $2 = \frac{1}{2} \mid \left(\frac{3k-5}{k+1}\right).(5-(-2)) + 1.\left((-2) - \frac{5k+1}{k+1}\right) + 7.\left(\frac{5k+1}{k+1} - 5\right) \mid$

$\Rightarrow k = 7$ or $k = \frac{31}{9}$.

110. Vertices: $A = (6,3), B = (-3,5), C = (4,-2), D = (x, 3x)$.

Area formula: $\Delta ABC = \left(\frac{1}{2}\right)|6.(5-(-2)) + (-3).(-2-3) + 4.(3-5)| = \left(\frac{1}{2}\right)|6.7 + (-3).(-5) + 4.(-2)| = \left(\frac{1}{2}\right)|42 + 15 - 8| = \left(\frac{1}{2}\right).49 = 24.5$

Area $BCD = \left(\frac{1}{2}\right)|(-3).(-2-3x) + 4.(3x-5) + x.(5-(-2))| = \left(\frac{1}{2}\right)|(-3).(-2-3x) + 4.(3x-5) + x.7|\ = \left(\frac{1}{2}\right)|6 + 9x + 12x - 20 + 7x| = \left(\frac{1}{2}\right)|28x - 14| = |14x - 7|$

Given $\Delta ABC = 2\Delta BCD \Rightarrow 24.5 = 2.|14x - 7| \Rightarrow |14x - 7| = 12.25$

Solving: $14x - 7 = 12.25 \Rightarrow 14x = 19.25 \Rightarrow x = \frac{19.25}{14} = \frac{77}{56}$

$14x - 7 = -12.25 \Rightarrow 14x = -5.25 \Rightarrow x = -\frac{5.25}{14} = -\frac{21}{56} = -\frac{3}{8}$.

111. Vertices: $A = (1, 2), B = (-5, 6), C = (7, -4), D = (h, -2)$.

$\Delta = 0 \Rightarrow 1.(6 - (-4)) + (-5).(-4 - (-2)) + 7.(-2 - 2) + h.(2 - 6) = 0$

$\Rightarrow h = 3$

112. Vertices: $A = (3, 4), B = (-4, 3), C = (8, 6)$.

Area: $\Delta = \left(\frac{1}{2}\right)|3.(3 - 6) + (-4).(6 - 4) + 8.(4 - 3)| = \left(\frac{1}{2}\right)|3.(-3) + (-4).2 + 8.1| = \left(\frac{1}{2}\right)| - 9 - 8 + 8| = \left(\frac{1}{2}\right).(-9) = \frac{9}{2}$

Length of perpendicular from $A$ on $BC$: $h = \frac{2.\Delta}{B}C$

$BC = \sqrt{(8 + 4)^2 + (6 - 3)^2} = \sqrt{12^2 + 3^2} = \sqrt{144 + 9} = \sqrt{153}$

$h = \frac{2.\left(\frac{9}{2}\right)}{\sqrt{153}} = \frac{9}{\sqrt{153}}$.

113. Centroid $G = \left(\frac{2}{3}, 2\right)$, vertices $A = (2, 3), B = (-1, 2), C = (x, y)$.

Centroid formula: $G = \left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3}\right) \Rightarrow \left(\frac{2}{3}, 2\right) = \left(\frac{2 - 1 + x}{3}, \frac{3 + 2 + y}{3}\right)$

Solve for $C$: $\frac{2 - 1 + x}{3} = \frac{2}{3} \Rightarrow x = 1$ and $\frac{3 + 2 + y}{3} = 2 \Rightarrow y = -1$

Area: $\Delta = \left(\frac{1}{2}\right)|\ 2.(2 - (-1)) + (-1).(-1 - 3) + 1.(3 - 2)\ | = \left(\frac{1}{2}\right)|2.3 + (-1).(-4) + 1.1| = \left(\frac{1}{2}\right)|(6 + 4 + 1)| = \frac{11}{2}$.

114. Vertices: $A = (3, 1), B = (1, -3), C = (x, y)$, centroid $G = \left(\frac{3 + 1 + x}{3}, \frac{1 - 3 + y}{3}\right)$ lies on $x$-axis $\implies$ $y$-coordinate = 0

$\frac{1 - 3 + y}{3} = 0 \Rightarrow y = 2$

Area formula: $\Delta = 3 = \left(\frac{1}{2}\right)|\ 3.(-3 - 2) + 1.(2 - 1) + x.(1 - (-3))\ | \Rightarrow x = 5$, or $-14 + 4x = -6 \Rightarrow x = 2$

So $C = (5, 2)$ or $C = (2, 2)$

115. Given: $A = (3, 4), B = (5, -2), P = (x, y)$ such that $PA = PB$

Perpendicular bisector condition: $(x - 3)^2 + (y - 4)^2 = (x - 5)^2 + (y + 2)^2$

Simplifying: $x^2 - 6x + 9 + y^2 - 8y + 16 = x^2 - 10x + 25 + y^2 + 4y + 4$

$4x - 12y = -12 \Rightarrow x - 3y = -3$

Area formula: $\Delta PAB = 10 = \frac{1}{2})|\ 3(-2 - y) + 5(y - 4) + x(4 - (-2))\ |$

Simplifying: $20 = |-6 - 3y + 5y - 20 + 6x\ | = |6x + 2y - 26|$

Solve system: $x - 3y = -3$ and $6x + 2y - 26 = \pm 20$

**Case I**: $6x + 2y - 26 = 20 \Rightarrow 6x + 2y = 46 \Rightarrow 3x + y = 23$

Solving with $x - 3y = -3 \Rightarrow x = \frac{17}{2}, y = \frac{11}{3}$

**Case II**: $6x + 2y - 26 = -20 \Rightarrow 6x + 2y = 6 \Rightarrow 3x + y = 3$

Solving with $x - 3y = -3 \Rightarrow x = -\frac{3}{2}, y = -\frac{1}{2}$.

116. Let the points be $A = (a, b + c), B = (b, c + a), C = (c, a + b)$.

Area of triangle: $\Delta = \left(\frac{1}{2}\right) \mid a((c+a)-(a+b))+b((a+b)-(b+c))+c((b+c)-(c+a)) \mid$

Simplifying each term: $((c+a)-(a+b)) = c-b$, $((a+b)-(b+c)) = a-c$, $((b+c)-(c+a)) = b-a$

So $\Delta = \frac{1}{2} \mid a(c-b)+b(a-c)+c(b-a) \mid$

Expanding: $a(c-b)+b(a-c)+c(b-a) = ac-ab+ab-bc+bc-ac = 0$

Hence, $\Delta = 0$ i.e. points are collinear.

117. Let the points be $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$.

If points are collinear, $\Delta = 0 : \Delta = \frac{1}{2} \mid x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2) \mid = 0$

Divide both sides by $x_1 x_2 x_3 n = 0$: $0 = \frac{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)}{x_1 x_2 x_3}$

Splitting terms: $0 = \frac{y_2-y_3}{x_2 x_3} + \frac{y_3-y_1}{x_3 x_1} + \frac{y_1-y_2}{x_1 x_2}$

Hence proved.

118. Points: $A = (a, b), B = (a_1, b_1), C = (a-a_1, b-b_1)$.

Collinear $\Rightarrow \Delta = 0 \Rightarrow 0 = \left(\frac{1}{2}\right) \mid a(b_1-(b-b_1))+a_1((b-b_1)-b)+(a-a_1)((b-b_1)-b_1) \mid$

Simplifying: $a_1 b = a b_1 \Rightarrow \frac{a}{a_1} = \frac{b}{b_1}$.

119. Points: $A = (a, 0), B = (0, b), C = (1, 1)$.

Collinear $\Rightarrow \Delta = 0 \Rightarrow 0 = \left(\frac{1}{2}\right) \mid a(b-1)+0(1-0)+1(0-b) \mid$

Simplify: $0 = a(b-1)-b \Rightarrow ab-a-b = 0 \Rightarrow \frac{1}{a}+\frac{1}{b} = 1$

Hence, points are collinear if $\frac{1}{a}+\frac{1}{b} = 1$.

120. Let the points be $A = (-4, -1), B = (-2, -4), C = (4, 0), D = (2, 3)$.

$AB = \sqrt{(-2+4)^2+(-4+1)^2} = \sqrt{2^2+(-3)^2} = \sqrt{4+9} = \sqrt{13}$

$BC = \sqrt{(4+2)^2+(0+4)^2} = \sqrt{6^2+4^2} = \sqrt{36+16} = \sqrt{52}$

$CD = \sqrt{(2-4)^2+(3-0)^2} = \sqrt{(-2)^2+3^2} = \sqrt{4+9} = \sqrt{13}$

$DA = \sqrt{(-4-2)^2+(-1-3)^2} = \sqrt{(-6)^2+(-4)^2} = \sqrt{36+16} = \sqrt{52}$

Opposite sides equal: $AB = CD, BC = DA$.

Diagonals: $AC = \sqrt{(4+4)^2+(0+1)^2} = \sqrt{8^2+1^2} = \sqrt{65}$

$BD = \sqrt{(2+2)^2+(3+4)^2} = \sqrt{4^2+7^2} = \sqrt{16+49} = \sqrt{65}$

Diagonals equal $\Rightarrow$ all angles are right angles.

121. Let the three consecutive vertices be $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$, and let $D = (x_4, y_4)$ be the fourth vertex.

In a parallelogram, the diagonals bisect each other $\Rightarrow \frac{x_1+x_3}{2} = \frac{x_2+x_4}{2}$ and $\frac{y_1+y_3}{2} = \frac{y_2+y_4}{2}$

Solving for $D$: $x_4 = x_1 + x_3 - x_2$ and $y_4 = y_1 + y_3 - y_2$

Hence, the fourth vertex is $D = (x_1 + x_3 - x_2, y_1 + y_3 - y_2)$.

122. Let $B = (0, 0), C = (c, 0)$ and $D$ be the midpoint of $BC \Rightarrow D = \left(\frac{c}{2}, 0\right)$. Let $A = (x, y)$.

$AB^2 = (x-0)^2 + (y-0)^2 = x^2 + y^2$

$AC^2 = (x-c)^2 + (y-0)^2 = (x-c)^2 + y^2$

$BD^2 = \left(\frac{c}{2} - 0\right)^2 + (0-0)^2 = \frac{c^2}{4}$

$AD^2 = \left(x - \frac{c}{2}\right)^2 + (y-0)^2 = \left(x - \frac{c}{2}\right)^2 + y^2$

Left-hand side: $AB^2 + AC^2 = x^2 + y^2 + (x-c)^2 + y^2 = 2x^2 - 2cx + c^2 + 2y^2$

Right-hand side: $2(AD^2 + BD^2) = 2\left(\left(x - \frac{c}{2}\right)^2 + y^2 + \frac{c^2}{4}\right) = 2\left(x^2 - cx + \frac{c^2}{4} + y^2 + \frac{c^2}{4}\right) = 2x^2 - 2cx + c^2 + 2y^2$

Hence, $AB^2 + AC^2 = 2(AD^2 + BD^2)$.

123. Let $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ and $G$ be the centroid: $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Let $O = (x_0, y_0)$.

Using distance formula: $OA^2 + OB^2 + OC^2 = (x_0 - x_1)^2 + (y_0 - y_1)^2 + (x_0 - x_2)^2 + (y_0 - y_2)^2 + (x_0 - x_3)^2 + (y_0 - y_3)^2$

Grouping terms: $= 3x_0^2 - 2x_0(x_1 + x_2 + x_3) + (x_1^2 + x_2^2 + x_3^2) + 3y_0^2 - 2y_0(y_1 + y_2 + y_3) + (y_1^2 + y_2^2 + y_3^2)$

Since $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$, rewriting: $OA^2 + OB^2 + OC^2 = GA^2 + GB^2 + GC^2 + 3GO^2$.

124. Let $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$, and let $D, E, F$ be the midpoints of $BC, CA, AB$ respectively:

$D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$, $E = \left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right)$, $F = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

$\Delta_{ABC} = \frac{1}{2} | \; x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \; |$

$\Delta_{DEF} = \frac{1}{2} | \; D_{x(E_y - F_y)} + E_{x(F_y - D_y)} + F_{x(D_y - E_y)} \; |$

Substituting midpoints: $\Delta_{DEF} = \frac{1}{2}.\left(\frac{1}{2}\right)^2 \mid x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \mid = \frac{1}{4}\Delta_{ABC}$

Hence, $\Delta_{ABC} = 4\Delta_{DEF}$.

125. Let $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ be the vertices of a triangle. Let $D$ and $E$ be the midpoints of $AB$ and $AC$:

$D = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, $E = \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$.

Length of $DE$: $DE^2 = \left(\frac{x_1+x_3}{2} - \frac{x_1+x_2}{2}\right)^2 + \left(\frac{y_1+y_3}{2} - \frac{y_1+y_2}{2}\right)^2$

$= \left(\frac{x_3-x_2}{2}\right)^2 + \left(\frac{y_3-y_2}{2}\right)^2 = \left(\frac{1}{4}\right)\left((x_3 - x_2)^2 + (y_3 - y_2)^2\right)$

But $BC^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2 \Rightarrow DE = \frac{1}{2}BC$.

126. Let $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$.

Let $P$ divide $BC$ in ratio $k : 1 \Rightarrow P = \left(\frac{kx_3+x_2}{k+1}, \frac{ky_3+y_2}{k+1}\right)$

Let $Q$ divide $CA$ in ratio $k : 1 \Rightarrow Q = \left(\frac{kx_1+x_3}{k+1}, \frac{ky_1+y_3}{k+1}\right)$

Let $R$ divide $AB$ in ratio $k : 1 \Rightarrow R = \left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}\right)$

Centroid of $ABC$: $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Centroid of $PQR$: $G' = \left(\frac{\frac{kx_3+x_2}{k+1}+\frac{kx_1+x_3}{k+1}+\frac{kx_2+x_1}{k+1}}{3}, \frac{\frac{ky_3+y_2}{k+1}+\frac{ky_1+y_3}{k+1}+\frac{ky_2+y_1}{k+1}}{3}\right)$

Simplifying $G'$ gives $G' = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) = G$

Hence, the centroids coincide.

127. Let $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$. Let $D, E, F$ be the midpoints of $BC, CA, AB$ respectively.

$D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$, $E = \left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right)$, and $F = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Medians: $AD, BE, CF$

Using distance formula and simplifying: $AD^2 = \frac{1}{4}[2(AB^2 + AC^2) - BC^2]$

$BE^2 = \frac{1}{4}[2(BC^2 + AB^2) - CA^2]$

$CF^2 = \frac{1}{4}[2(CA^2 + BC^2) - AB^2]$

$\Rightarrow AD^2 + BE^2 + CF^2 = \frac{1}{4}[3(AB^2 + BC^2 + CA^2)]$

$\Rightarrow 4(AD^2 + BE^2 + CF^2) = 3(AB^2 + BC^2 + CA^2)$.

128. Let $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$. Centroid $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

$GA^2 + GB^2 + GC^2 = \left(\frac{1}{3}\right)\left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 + (y_1 - y_2)^2 + (y_2 - y_3)^2 + (y_3 - y_1)^2\right]$

But, $AB^2 + BC^2 + CA^2 = (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 + (y_1 - y_2)^2 + (y_2 - y_3)^2 + (y_3 - y_1)^2$

Hence, $AB^2 + BC^2 + CA^2 = 3(GA^2 + GB^2 + GC^2)$.

129. Let $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$ and centroid $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

Area of $\triangle ABG$: $\Delta ABG = \frac{1}{2} \mid x_1(y_2 - y_G) + x_2(y_G - y_1) + x_{G(y_1-y_2)} \mid$

Substitute $x_G, y_G$ and simplify: $\Delta ABG = \frac{1}{3}\Delta ABC$

Similarly, $\Delta BCG = \frac{1}{3}\Delta ABC$ and $\Delta CAG = \frac{1}{3}\Delta ABC$

Hence, $\Delta ABG = \Delta BCG = \Delta CAG$.

130. Let the right angled triangle be right angled at $A$. Take $A = (0,0), B = (a,0), C = (0,b)$.

Midpoint of hypotenuse $BC$: $D = \left(\frac{a+0}{2}, \frac{0+b}{2}\right) = \left(\frac{a}{2}, \frac{b}{2}\right)$

Distances: $DA^2 = \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \frac{a^2+b^2}{4}$

$DB^2 = \left(\frac{a}{2} - a\right)^2 + \left(\frac{b}{2} - 0\right)^2 = \frac{a^2+b^2}{4}$

$DC^2 = \left(\frac{a}{2} - 0\right)^2 + \left(\frac{b}{2} - b\right)^2 = \frac{a^2+b^2}{4}$

Thus, $DA = DB = DC$.

131. Let $A = (a_1, b_1), B = (a_2, b_2), C = (a_3, b_3)$.

$D$ divides $AB$ in ratio $\lambda : \mu$: $D = \left(\frac{\lambda.a_2+\mu.a_1}{\lambda+\mu}, \frac{\lambda.b_2+\mu.b_1}{\lambda+\mu}\right)$

$E$ divides $DC$ in ratio $\mu : (\lambda+\mu)$: $E = \left(\frac{\mu.a_3+(\lambda+\mu).\frac{\lambda.a_2+\mu.a_1}{\lambda+\mu}}{\lambda+2\mu}, \frac{\mu.b_3+(\lambda+\mu).\frac{\lambda.b_2+\mu.b_1}{\lambda+\mu}}{\lambda+2\mu}\right)$

Simplifying: $E = \left(\frac{\lambda.a_2+\mu.a_1+\mu.a_3}{\lambda+2\mu}, \frac{\lambda.b_2+\mu.b_1+\mu.b_3}{\lambda+2\mu}\right)$.

132. Given roots: $\alpha+\beta = -2\frac{h}{a}$, $\alpha.\beta = \frac{b}{a}$ and $\gamma+\delta = -2\frac{h'}{a}$, $\gamma.\delta = \frac{b'}{a}$

Ratios in which $C$ and $D$ divide $AB$: $\frac{AC}{CB} = \frac{\gamma-\alpha}{\beta-\gamma}$ and $\frac{AD}{DB} = \frac{\delta-\alpha}{\beta-\delta}$

Sum: $S = \frac{\gamma-\alpha}{\beta-\gamma} + \frac{\delta-\alpha}{\beta-\delta}$

Simplify numerator: $S = \frac{(\gamma-\alpha)(\beta-\delta)+(\delta-\alpha)(\beta-\gamma)}{(\beta-\gamma)(\beta-\delta)}$

Numerator expands to: $\beta(\gamma+\delta) - \alpha(\gamma+\delta) - 2\gamma.\delta + 2\alpha.\beta$

Substitute root relations: $= \beta\left(-2\frac{h'}{a}\right) - \alpha\left(-2\frac{h'}{a}\right) - 2\left(\frac{b'}{a}\right) + 2\left(\frac{b}{a}\right) = \left(2\frac{h'}{a}\right)(\alpha - \beta) + 2\frac{b-b'}{a}$

Using condition $a.b' + a'.b = 2h.h'$ gives numerator $= 0$

Hence: $S = 0$

Thus, the sum of the ratios in which $C$ and $D$ divide $AB$ is zero.

133. Given $A = (1,-2), B = (2,5)$ and $O = (0,0)$.

$OC = 2OA \Rightarrow C = (2.1, 2.(-2)) = (2,-4)$ and $OD = 2OB \Rightarrow D = (2.2, 2.5) = (4,10)$

Distance $CD$: $CD = \sqrt{(4-2)^2 + (10-(-4))^2} = \sqrt{2^2+14^2} = \sqrt{4+196} = \sqrt{200} = 10\sqrt{2}$

So, $CD = 10\sqrt{2}$.

134. Given $A = (2,1), B = (3,-1)$ and $C = (x,y)$ with $y = x+9$.

Centroid $G = \left(\frac{2+3+x}{3}, \frac{1-1+y}{3}\right)$ lies on $y$-axis => x-coordinate of $G = 0$: $\frac{5+x}{3} = 0 \Rightarrow$ $x = -5$

Then $y = x + 9 = 4$ => $C = (-5, 4)$.

Centroid: $G = \left(\frac{2+3-5}{3}, \frac{1-1+4}{3}\right) = \left(0, \frac{4}{3}\right)$

So, $C = (-5, 4)$ and the centroid is $\left(0, \frac{4}{3}\right)$.

135. Let the vertices of the triangle be $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$.

Centroid: $(x, y) = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

Circumcenter: $(\alpha, \beta)$

Orthocenter: $(p, q)$

In coordinate geometry, the orthocenter satisfies: $p = x_1 + x_2 + x_3 - 2\alpha$ and $q = y_1 + y_2 + y_3 - 2\beta$

Multiplying centroid coordinates by 3: $3x = x_1 + x_2 + x_3$ and $3y = y_1 + y_2 + y_3$

Substitute: $3x = 2\alpha + p$ and $3y = 2\beta + q$

Hence proved.

136. Vertices: $A = (2, 3), B = (3, 4), C = (6, 8)$

Centroid $G$: $G = \left(\frac{2+3+6}{3}, \frac{3+4+8}{3}\right) = \left(\frac{11}{3}, 5\right)$

Circumcenter $(\alpha, \beta)$: Using the perpendicular bisector formula: the circumcenter is the intersection of the perpendicular bisectors of any two sides.

Equation of perpendicular bisector of $AB$: Midpoint $M_1 = \left(\frac{2+3}{2}, \frac{3+4}{2}\right) = \left(\frac{5}{2}, \frac{7}{2}\right)$

If $C = (6, 8)$, line AC: slope not needed, solve using: $(x - \alpha)^2 + (y - \beta)^2 = (x_C - \alpha)^2 + (y_C - \beta)^2$

$(2 - \alpha)^2 + (3 - \beta)^2 = (3 - \alpha)^2 + (4 - \beta)^2 \Rightarrow (2 - \alpha)^2 + (3 - \beta)^2 = (6 - \alpha)^2 + (8 - \beta)^2$

Solve these two equations simultaneously: $\alpha = \frac{9}{2}, \beta = \frac{3}{2}$

Orthocenter $(p, q)$ using formula $3x = 2\alpha + p, 3y = 2\beta + q$: $p = 3\frac{11}{3} - 2\frac{9}{2} = 2$ and $q = 35 - 2\frac{3}{2} = 12$.

137. Let $A\left(\alpha, \frac{1}{\alpha}\right), B\left(\beta, \frac{1}{\beta}\right), C\left(\gamma, \frac{1}{\gamma}\right)$ be the vertices of a triangle.

$\alpha$ and $\beta$ are roots of $x^2 - 6p_1x + 2 = 0$, so $\alpha + \beta = 6p_1$ and $\alpha\beta = 2$.

$\beta$ and $\gamma$ are roots of $x^2 - 6p_2x + 3 = 0$, so $\beta + \gamma = 6p_2$ and $\beta\gamma = 3$.

$\gamma$ and $\alpha$ are roots of $x^2 - 6p_3x + 6 = 0$, so $\gamma + \alpha = 6p_3$ and $\gamma\alpha = 6$.

Solving these gives $p_1 = 1$, $p_2 = 1$, $p_3 = 2$, $\alpha = 2$, $\beta = 1$, $\gamma = 3$.

The centroid of the triangle is $\left(\frac{\alpha+\beta+\gamma}{3}, \frac{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}{3}\right) = \left(\frac{2+1+3}{3}, \frac{\frac{1}{2}+1+\frac{1}{3}}{3}\right) = \left(2, \frac{11}{18}\right)$.

138. Let the vertices of the triangle be $A(\tan\alpha, \cot\alpha)$, $B(\tan\beta, \cot\beta)$, $C(\tan\gamma, \cot\gamma)$, where $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are roots of $x^3 - 3ax^2 + 3bx - 1 = 0$.

By Vieta's formulas, $\tan\alpha + \tan\beta + \tan\gamma = 3a$,

$\tan\alpha ... \tan\beta + \tan\beta ... \tan\gamma + \tan\gamma ... \tan\alpha = 3b$, and $\tan\alpha ... \tan\beta ... \tan\gamma = 1$.

The centroid of the triangle is $\left(\frac{\tan\alpha+\tan\beta+\tan\gamma}{3}, \frac{\cot\alpha+\cot\beta+\cot\gamma}{3}\right) = \left(\frac{3a}{3}, \frac{\cot\alpha+\cot\beta+\cot\gamma}{3}\right) = \left(a, \frac{\cot\alpha+\cot\beta+\cot\gamma}{3}\right)$.

Since $\cot\theta = \frac{1}{\tan\theta}$, we have $\cot\alpha + \cot\beta + \cot\gamma = \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} + \frac{1}{\tan\gamma} = \frac{\tan\beta.\tan\gamma+\tan\gamma.\tan\alpha+\tan\alpha.\tan\beta}{\tan\alpha.\tan\beta.\tan\gamma} = \frac{3b}{1} = 3b$.

Therefore, the centroid is $(a, b)$.

139. Let the forces $F_1 = 30$ N and $F_2 = 40$ N act at points $A(-3, -1)$ and $B(4, 6)$.

The resultant acts along the line joining $A$ and $B$ and its point of application $R(x, y)$ divides $AB$ in the inverse ratio of the forces: $AR : RB = F_2 : F_1 = 40 : 30 = 4 : 3$.

Using coordinate geometry, if a point divides the segment joining $(x_1, y_1)$ and $(x_2, y_2)$ in the ratio $m : n$, its coordinates are $\left(\frac{n(x_1)+m(x_2)}{m+n}, \frac{n(y_1)+m(y_2)}{m+n}\right)$.

Here $x_1 = -3$, $y_1 = -1$, $x_2 = 4$, $y_2 = 6$, $m = 4$, $n = 3$. Then

$x_R = \frac{3(-3)+4(4)}{3+4} = \frac{-9+16}{7} = 1$, and $y_R = \frac{3(-1)+4(6)}{3+4} = \frac{-3+24}{7} = 3$.

Thus, the point of application of the resultant force is $R(1, 3)$.

140. Let $A(-1, 3)$, $B(-2, 4)$, and the other vertices be $C(x_1, y_1)$, $D(x_2, y_2)$ with diagonals intersecting at $O$ on the positive $x$-axis.

For a parallelogram, diagonals bisect each other: $O = \left(\frac{-1+x_2}{2}, \frac{3+y_2}{2}\right) = \left(\frac{-2+x_1}{2}, \frac{4+y_1}{2}\right)$.

Let $O = (h, 0)$, then $y_1 = -4$, $y_2 = -3$, $x_1 = 2h + 2$, $x_2 = 2h + 1$.

Area formula: $|(x_1 + 1)(y_2 - 3) - (x_2 + 1)(y_1 - 3)| = 24$.

Substituting: $|(2h + 3)(-6) - (2h + 2)(-7)| = |2h - 4| = 24 \Rightarrow h = 14$.

Hence, $C(30, -4)$ and $D(29, -3)$.

141. Let $A(1, 2)$, $B(8, 4)$, $C(4, 10)$.

The point $P(x, y)$ such that triangles $PCB$, $PCA$, and $PAB$ have equal area is the centroid of $\triangle ABC$:

$P = \left(\frac{1+8+4}{3}, \frac{2+4+10}{3}\right) = \left(\frac{13}{3}, \frac{16}{3}\right)$.

142. Let $a, b, c$ be the $p$th, $q$th, $r$th terms of an H.P., so $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

Consider the points $X(bc, p)$, $Y(ca, q)$, $Z(ab, r)$.

Three points are collinear if $(bc)((q) - (r)) + (ca)((r) - (p)) + (ab)((p) - (q)) = 0$.

Since $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., this relation is satisfied.

Hence the points $(bc, p)$, $(ca, q)$, $(ab, r)$ are collinear.

143. Let $x_1, x_2, x_3$ be in A.P. and $y_1, y_2, y_3$ be in A.P.

Consider the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$.

Three points are collinear if the determinant $(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)) = 0$.

Since $x_1, x_2, x_3$ are in A.P., $x_2 - x_1 = x_3 - x_2$.

Since $y_1, y_2, y_3$ are in A.P., $y_2 - y_1 = y_3 - y_2$.

Then $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = x_1(-(y_2 - y_1)) + x_2((y_2 - y_1) + (y_2 - y_1)) + x_3(-(y_2 - y_1)) = 0$.

Hence the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are collinear.

144. Consider the points $(a, a^2)$, $(b, b^2)$, $(c, c^2)$.

Three points are collinear if $(a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)) = 0$.

Factor each difference of squares: $a(b - c)(b + c) + b(c - a)(c + a) + c(a - b)(a + b) = 0$.

Simplifying: $(a - b)(b - c)(c - a) \neq 0$ for distinct $a, b, c$.

Hence, the points $(a, a^2)$, $(b, b^2)$, $(c, c^2)$ are not collinear.

145. Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ be the vertices of a triangle, and let $M$ be the midpoint of $BC$, so the median through $A$ is the line joining $A$ and $M$.

Let $(x, y)$ be a point on this median. Then $M = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$ and $(x, y)$ lies on the line joining $A$ and $M$:

$\frac{y-y_1}{\frac{y_2+y_3}{2}-y_1} = \frac{x-x_1}{\frac{x_2+x_3}{2}-x_1}$.

Consider the determinants: $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$ and $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

The determinant $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}$ represents twice the signed area of triangle $AXB$, and $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ represents twice the signed area of triangle $AXC$.

Since $(x, y)$ lies on the median $AM$, triangles $AXB$ and $AXC$ have equal areas with opposite signs, so their determinants sum to zero:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

146. Let $A(2, -3)$, $B(3, -2)$, $C(x, y)$.

Area $= \frac{3}{2} \Rightarrow |2(-2-y)+3(y+3)+x(-3+2)|\frac{1}{2} = \frac{3}{2} \Rightarrow |x-y-5| = 3 \Rightarrow x-y = 8 \quad \text{or} \quad 2$.

Centroid $G = \left(\frac{2+3+x}{3}, \frac{-3-2+y}{3}\right)$ lies on $3x-y-8=0 \Rightarrow 3\left(\frac{5+x}{3}\right) - \left(\frac{-5+y}{3}\right) - 8 = 0 \Rightarrow 3x-y=-2$.

Solve with area condition: $x-y=8$ and $3x-y=-2 \Rightarrow x=-5, y=-13$.

Hence, $C(-5,-13)$.

147. Let $A(-2,5), B(4,-1), C(9,1), D(3,7)$.

Midpoint of $AC = \left(\frac{-2+9}{2}, \frac{5+1}{2}\right) = \left(\frac{7}{2}, 3\right)$, midpoint of $BD = \left(\frac{4+3}{2}, \frac{-1+7}{2}\right) = \left(\frac{7}{2}, 3\right)$, so $ABCD$ is a parallelogram.

Area $= |(-2.-1+4.1+9.7+3.5)-(5.4+-1.9+1.3+7.-2)|\frac{1}{2} = 42$.

$E$ divides $AC$ in ratio $2:1 \Rightarrow E = \left(\frac{16}{3}, \frac{7}{3}\right)$, midpoint of $BC$ is $F = \left(\frac{13}{2}, 0\right)$.

Determinant $(D \;\; E \;\; F) = 3.\left(\frac{7}{3}-0\right) - 7.\left(\frac{16}{3}-\frac{13}{2}\right) + 1.\left(\frac{16}{3.0} - \frac{\frac{7}{3.13}}{2}\right) = 0$, so $D, E, F$ are collinear.

148. Let the points be $A(-3,-1), B(2,-1), C(1,1), D(-2,1)$.

Compute slopes of opposite sides:

Slope of $AB = \frac{-1+1}{2+3} = \frac{0}{5} = 0$,

Slope of $CD = \frac{1-1}{1+2} = \frac{0}{3} = 0$.

Slope of $BC = \frac{1+1}{1-2} = \frac{2}{-}1 = -2$,

Slope of $AD = \frac{1+1}{-2+3} = \frac{2}{1} = 2$.

Since $AB \parallel CD$ and $BC \nparallel AD$, the quadrilateral is a trapezium.

149. Let the vertices of a triangle be $A(x_1,y_1), B(x_2,y_2), C(x_3,y_3)$ with integer coordinates.

The squared distance between two points is $AB^2 = (x_2-x_1)^2 + (y_2-y_1)^2$, similarly for $BC^2$ and $CA^2$.

If the triangle is equilateral, then $AB^2 = BC^2 = CA^2 = k^2$ for some integer $k^2$.

Consider the triangle modulo 2: the square of any integer is 0 or 1 modulo 4. The sum of two squares modulo 4 can be $0, 1, 2$, but never 3. In an equilateral triangle with integer coordinates, all three squared distances must be equal modulo 4.

It can be shown that no three distinct integer points satisfy $(x_2-x_1)^2 + (y_2-y_1)^2 = (x_3-x_2)^2 + (y_3-y_2)^2 = (x_1-x_3)^2 + (y_1-y_3)^2$ modulo 4. Hence, a triangle with integral coordinates cannot be equilateral.

# 2 Answers of Locus

1. We take the two perpendicular lines as axes of the coordinates. Let $(x, y)$ be any point satisfying the given condition. According to condition $x + y = a$.

   This is the relation which connects the coordinates of any point on the locus is the equation to the locus.

   In the next chapter we will study that it is an equation of a straight line.

2. Let $(x, y)$ be any position of the moving point. By given condition from the question we have

$$\{(x-a)^2 + y^2\} + \{(x+a)^2 + y^2\} = 2c^2 \tag{2.1}$$

$$\Rightarrow x^2 + y^2 = c^2 - a^2 \tag{2.2}$$

   This is the relation between coordinates of each and every point that satisfies the given condition, and is the equation to required locus.

   The equation tells us that the square of distance of the point $(x, y)$ from the origin is constant and equal to $c^2 - a^2$, and therefore, the locus of the point is a circle whose center is origin and radius is $\sqrt{c^2 - a^2}$.

3. Let the point be $P(x, y)$. According to question

   $\sqrt{(x+1)^2 + y^2} = 3\sqrt{x^2 + (y-2)^2} \Rightarrow (x+1)^2 + y^2 = 9(x^2 + (y-2)^2)$

   $\Rightarrow 8(x^2 + y^2) - 2x - 36y + 35 = 0.$

4. Let the point be $P(x, y)$. Given that $PA^2 - PB^2 = 2k^2$

   $\Rightarrow (x-a)^2 + y^2 - (x+a)^2 - y^2 = 2k^2 \Rightarrow -4ax = 2k^2 \Rightarrow 2ax + k^2 = 0.$

5. Given that $PA = nPB \Rightarrow PA^2 = n^2PB^2 \Rightarrow (x-a)^2 + y^2 = n^2[(x+a)^2 + y^2]$

   $\Rightarrow (n^2 - 1)(x^2 + y^2 + a^2) + 2ax(n^2 + 1) = 0.$

6. Given that $PA + PB = c \Rightarrow \sqrt{(x-a)^2 + y^2} + \sqrt{(x+a)^2 + y^2} = c$

   Squaring, $\Rightarrow (x-a)^2 + y^2 + (x+a)^2 + y^2 + 2\sqrt{(x-a)^2 + y^2}\sqrt{(x+a)^2 + y^2} = c^2$

   Squaring again, $\Rightarrow 4x^2(c^2 - 4a^2) + 4c^2y^2 = c^4 - 4a^2$

7. Given that $PB^2 + PC^2 = 2PA^2 \Rightarrow (x+a)^2 + y^2 + (x-c)^2 + y^2 = 2[(x-a)^2 + y^2]$

   Simplifying gives us $(6a - 2c)x = a^2 - c^2$.

8. Let $P(x, y)$ be the point in question. Its distance from $y$-axis would be $x$. Thus, according to question

   $(x-1)^2 + (y-2)^2 = x^2 \Rightarrow y^2 - 4y - 2x + 5 = 0.$

9. Let $P(x, y)$ be the point in question, then according to the question

   $(x-1)^2 + y^2 = x^2 + (y+2)^2 \Rightarrow -2x + 1 = 4y + 4 \Rightarrow 2x + 4y + 3 = 0.$

10. Let $P(x,y)$ be the point in question, then according to the question

    $(x-2)^2+(y-3)^2=(x-4)^2+(y-5)^2 \Rightarrow 4x+4y=28 \Rightarrow x+y=7.$

11. Let $P(x,y)$ be the point in question, then according to the question

    $(x-a-b)^2+(y-a+b)^2=(x-a+b)^2+(y-a-b)^2 \Rightarrow x=y.$

12. Distance from $x$-axis is $y$ and distance from $y$-axis is $x$, then according to the question $y=3x$.

13. $(x-a)^2+y^2=4^2.x^2 \Rightarrow 15x^2-y^2+2ax=a^2.$

14. According to question, $x+y=3$.

15. According to question, $x^2+(y-2)^2=4 \Rightarrow x^2+y^2-4y=0.$

16. According to question, $(x-3)^2+y^2=3^2\left[x^2+(y-2)^2\right] \Rightarrow 8x^2+8y^2+6x-36y+27=0.$

17. $y=\frac{1}{2}\sqrt{x^2+y^2} \Rightarrow x^2=3y^2.$

18. Let the fixed straight line be parallel to the x-axis, and let the fixed point be the origin $O(0,0)$.

    Since the fixed point is at a perpendicular distance $a$ from the line, and the line is parallel to the $x$-axis, its equation must be:

    $y=a$

    We consider moving point as $P(x,y)$.

    Its distance from origin is $\sqrt{x^2+y^2}$. Its distance from fixed line is $|y-a|$.

    Thus, $x^2+y^2=(y-a)^2 \Rightarrow x^2+2ay=a^2.$

19. From given condition $\left(x^2+y^2\right)=\frac{1}{4}(y-a)^2 \Rightarrow 4x^2+3y^2+2ay=a^2$ and $\frac{1}{4}(x^2+y^2)=(y-a)^2 \Rightarrow x^2-3y^2+8ay=4a^2.$

20. Let $P\equiv(x,y)$, then $x=a\cos\theta, y=b\sin\theta$

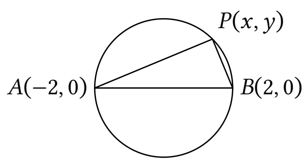
    $\cos\theta=\frac{x}{a}, \sin\theta=\frac{y}{b} \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

21. Let $P(x,y)$ be the required point, then according to the question

    $\sqrt{x^2+(y-2)^2}+\sqrt{x^2+(y+2)^2}=6 \Rightarrow 9x^2+5y^2=45.$

22. Let $AB=2a$, where $A$ is $(-a,0)$ and $B$ is $(a,0)$. Let $P(x,y)$ be the moving point. Then according to the question

    $AP^2+PB^2=AB^2 \Rightarrow (x+a)^2+y^2+(x-a)^2+y^2=4a^2 \Rightarrow x^2+y^2=a^2.$



**Tip**: The obtained equation is a circle(we will see this in later chapters) which is evident because from diameters the angle on any point on the perimeter is a right angle.

23. The diagram is given below:

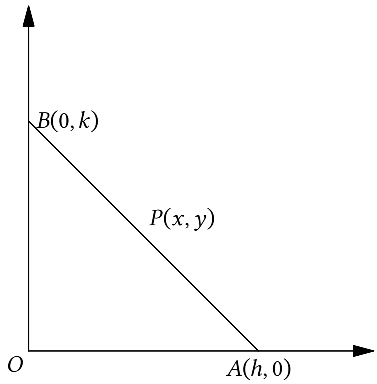
    Let $A \equiv (h, 0), B \equiv (0, k)$ and $P \equiv (x, y)$. Given $AP = a, PB = b$.

    Now $\frac{AP}{PB} = \frac{a}{b}$, hence, $P$ divides $AB$ in the ration $a : b$.

    Thus, $x = \frac{a.0+b.h}{a+b} = b\frac{h}{a+b}, y = \frac{a.k+b.0}{a+b} = a\frac{k}{a+b}$

    Also, $AB^2 = OA^2 + OB^2 \Rightarrow (a+b)^2 = h^2 + k^2 = \frac{(a+b)^2x^2}{b^2} + \frac{(a+b)^2y^2}{a^2}$

    $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$




24. Let $A \equiv (x_1, y_1)$. Let the moving point be $P(\alpha, \beta)$ then according to the question $A(x_1, y_1)$ lies on the curve $y^2 = 8x$.

    $\Rightarrow y_1^2 = 8x_1$. Also, since $P$ is the mid-point of $OA$,

    $\alpha = \frac{x_1}{2} \Rightarrow x_1 = 2\alpha$ and $\beta = \frac{y_1}{2} \Rightarrow 2\beta$

    Thus, $4\beta^2 = 8.2\alpha \Rightarrow \beta^2 = 4\alpha$. Thus, locus of point $P$ is $y^2 = 4x$.

25. Let $f(x, y) = x^2 + y^2 - 2x + 1$. $f(2, -5) = 26 \neq 0$.

    Thus, the point $(2, -5)$ does not lie on the given curve.

26. Since the given equation represent the identical curves the ratio of coefficients of terms must be equal. Thus,

    $m_1\frac{m_2}{a} = -\frac{m_1+m_2}{2}h = \frac{1}{b} \Rightarrow m_1m - 2 = \frac{a}{b}, m_1 + m_2 = 2\frac{h}{b}.$

27. Let $(x, y) \equiv (at^2, 2at)$. Then $y^2 = 4ax$.

28. Let $(x, y) \equiv \left(t + \frac{1}{t}, t - \frac{1}{t}\right)$, then $x^2 - y^2 = 4$.

29. Let $(x, y) \equiv (\cos\theta + \sin\theta, \cos\theta - \sin\theta)$, then $x^2 + y^2 = 2$.

30. Centroid will be given by $\left(\frac{\cos\theta+\sin\theta+1}{3}, \frac{\sin\theta+\cos\theta+2}{3}\right)$. Let it be $(x, y)$.

    Let $\cos\theta + \sin\theta = S$, then $x = \frac{S+1}{3} \Rightarrow 3x = S + 1$ and $3y = S + 2$.

Subtracting we get $3x - 3y + 1 = 0$.

31. Let $P(u\cos\alpha.t, u\sin\alpha.t - kt^2) \equiv (x, y)$ be the moving point. Then

    $x = u\cos\alpha.t \Rightarrow t = \frac{x}{u\cos\alpha}$ and $y = u\sin\alpha.t - kt^2$

    Substituting for $t$ in $y$ we have

    $y = x\tan\alpha - \frac{kx^2}{u^2\cos^2\alpha}$.

32. Let $P(x, y)$ be the point whose locus is to be determined, then according to the question

    $(x+2)^2 + (y-3)^2 = 3^2[(x^2 + (y-3)^2)] \Rightarrow 8x^2 - 4x - 4 + 8(y-3)^2 = 0$

    $\Rightarrow 2x^2 + 2y^2 - x - 12y + 17 = 0$.

33. Let $P(x, y)$ be the point whose locus is to be determined, then according to the question

    $4PA^2 = 9PB^2 \Rightarrow 4[(x+5)^2 + (y-3)^2] = 9[(x-2)^2 + (y-4)^2]$

    $\Rightarrow 5x^2 + 5y^2 - 76x - 48y + 44 = 0$.

34. Let $P(x, y)$ be the point whose locus is to be determined, then according to the question

    $PS = \sqrt{(x-4)^2 + y^2}$, $PM = x$ and $PS = PM \Rightarrow (x-4)^2 + y^2 = x^2$

    $\Rightarrow y^2 - 8x + 16 = 0$.

35. Let the given points be $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $P(x, y)$ be a point such that $PA = PB$.

    $PA^2 = (x - x_1)^2 + (y - y_1)^2$ and $PB^2 = (x - x_2)^2 + (y - y_2)^2$

    Since $PA = PB \Rightarrow (x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2$

    $x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 = x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2$

    $-2xx_1 - 2yy_1 + x_1^2 + y_1^2 = -2xx_2 - 2yy_2 + x_2^2 + y_2^2$

    Rearranging, $2x(x_2 - x_1) + 2y(y_2 - y_1) = x_2^2 + y_2^2 - x_1^2 - y_1^2$

    This is a linear equation in $x$ and $y$, hence it represents a straight line.

    Now let $M$ be the midpoint of $AB$. Then $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

    Substituting the coordinates of $M$ into the equation satisfies it, so the line passes through the midpoint of $AB$.

    Now we consider triangles $PAM$ and $PBM$ and we see that $PA = PB, PM = PM, AM = BM$ so both the triangles are congruent. We also find that $PA^2 = PM^2 + AM^2$, thus, the triangles are right angled as well at $M$. Thus, we have proven that the locus bisects $AB$ at right angle.

36. Distance of $P(x, y)$ from $(a, 0)$ is $\sqrt{(x-a)^2 + y^2} = a + x$

    Sqauring gives us, $\Rightarrow (x-a)^2 + y^2 = (x+a)^2 \Rightarrow y^2 = 4ax$.

37. Let $A(1, 2)$, $B(-2, 3)$ and $P(x, y)$.

The area of triangle $PAB$ is given by

$\Delta = \frac{1}{2}\,|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$

Substituting $A(1,2)$, $B(-2,3)$ and $P(x,y)$,

$\Delta = \frac{1}{2}\mid x(2-3) + 1(3-y) + (-2)(y-2)\mid$

$= \frac{1}{2}\mid x(-1) + 3 - y - 2y + 4\mid$

$= \frac{1}{2}\mid -x - 3y + 7\mid = 9$

$\mid -x - 3y + 7\mid = 18$

$x + 3y = -11$ or $x + 3y = 25$

38. Let $A(1,2)$ and let $Q(x,y)$ be a variable point on the curve $x^2 + y^2 + x + y = 0$.

Let $P(h,k)$ be the midpoint of $AQ$. By the midpoint formula,

$h = \frac{1+x}{2}, k = (2+y)(2)$.

Hence, $x = 2h - 1, y = 2k - 2$

Since $Q$ lies on the curve, $(2h-1)^2 + (2k-2)^2 + (2h-1) + (2k-2) = 0$

$\Rightarrow 4h^2 - 4h + 1 + 4k^2 - 8k + 4 + 2h - 1 + 2k - 2 = 0$

$\Rightarrow 2h^2 + 2k^2 - h - 3k + 1 = 0$

So the locus is $2x^2 + 2y^2 - x - 3y + 1 = 0$.

39. If $P(x,y)$ divides $AQ$ internally in the ratio $3:1$, then by section formula,

$x = \frac{3.3\cos\theta + 1.2}{4}$, $y = \frac{3.2\sin\theta + 1.3}{4}$.

$x = \frac{9\cos\theta + 2}{4}$, $y = \frac{6\sin\theta + 3}{4}$.

Rearranging, $4x - 2 = 9\cos\theta$, $4y - 3 = 6\sin\theta$.

Thus, $\cos\theta = \frac{4x-2}{9}$, $\sin\theta = \frac{4y-3}{6}$.

Using $\cos^2\theta + \sin^2\theta = 1$,

$\left(\frac{4x-2}{9}\right)^2 + \left(\frac{4y-3}{6}\right)^2 = 1$.

Hence the locus of $P$ is $\frac{(4x-2)^2}{81} + \frac{(4y-3)^2}{36} = 1$.

40. Let a variable line through $A(6,-8)$ cut the $x$-axis at $Q(a,0)$.

Let $P(x,y)$ be the midpoint of $AQ$.

By midpoint formula, $x = \frac{6+a}{2}$, $y = \frac{-8+0}{2}$.

So, $x = \frac{6+a}{2}$, $y = -4$.

From the first equation, $a = 2x - 6$.

Since $a$ is arbitrary (any real number), $x$ can take any real value, while $y$ is constant.

Hence the locus of $P$ is $y = -4$.

# 3 Answers of Straight Lines

1. Since the intercept and angle with $x$-axis is given, therefore, we can represent this line using slope intercept form i.e. $y = mx + c$.

   $m = \tan 45^\circ = 1$ and $c = 5$. Therefore, the required equation is $y = x + 5$.

2. Let the interecepts be $a, -a$ for the two axes. So we can represent the line as $\frac{x}{a} - \frac{y}{a} = 1$. Since it passes through $(2, -3) \Rightarrow \frac{2}{a} - \frac{-3}{a} = 1 \Rightarrow a = 5$.

   Thus, the required equation is $x - y = 5$.

3. Let the required straight line be $\frac{x}{a} + \frac{y}{b} = 1$, which will meet the axes at $(a, 0)$ and $(0, b)$.

   The coordinate of the point dividing the line joining these points in the ratio $1 : 2$ is $\frac{2.a+1.0}{2+1}$ and $\frac{2.0+1.b}{2+1}$ i.e. $\frac{2a}{3}$ and $\frac{b}{3}$.

   Thus, $\frac{2a}{3} = -5 \Rightarrow a = -\frac{15}{2}$ and $4 = \frac{b}{3} \Rightarrow b = 12$.

   Thus, the required equationis $\frac{x}{-\frac{15}{2}} + \frac{y}{12} = 1 \Rightarrow 5y - 8x = 60$.

4. Comparing the equation with $ax + by + c = 0$ we have $a = 1, b = \sqrt{3}, c = 1$

   $\sqrt{a^2 + b^2} = \sqrt{1 + 3} = 2$

   Dividing the given equation by 2 gives us
   $\frac{x}{2} + y\frac{\sqrt{3}}{2} + \frac{7}{2} = 0 \Rightarrow x \cos 240^\circ + y \sin 240^\circ = \frac{7}{2}$.

5. The equation of straight in two-point form is given by $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

   So the equation becomes $y - 3 = \frac{-2-3}{4+1}(x + 1) \Rightarrow x + y = 2$.

6. Since the intercept and slope is given we can represent it as $y = mx + c$. Given that $c = 1$ and $m = \tan 45^\circ = 1$.

   Therefore, the required equation is $y = x + 1$.

7. Since the intercept and slope is given we can represent it as $y = mx + c$. Given that $c = -5$. Since the line is equally inclined to the axes so angle of inclination is $\theta = 45^\circ$.

   Thus, $m = \tan \theta = 1$, and thus, the required equation is $y = x - 5$.

8. Since the intercept and slope is given we can represent it as $y = mx + c$. Given that $c = -2$. Since the line is inclined at $30^\circ$ to $OX$ so $m = \tan 30^\circ = \frac{1}{\sqrt{3}}$.

   Thus, required equation is $y = \frac{x}{\sqrt{3}} - 2 \Rightarrow \sqrt{3}y = x - 2\sqrt{3}$.

9. Since the intercept and slope is given we can represent it as $y = mx + c$. Given that $c = -3$. Since the line is inclined at an angle $\tan^{-1} \frac{3}{5}$, therefore $m = \tan \tan^{-1} \frac{3}{5} = \frac{3}{5}$.

   Thus, required equation is $y = \frac{3}{5}x - 3 \Rightarrow 5y = 3x - 9$.

10. Since intercepts are given so we can use the intercept form i.e. $\frac{x}{a} + \frac{y}{b} = 1$. Given that $a = 2$ and $b = 3$.

    Therefore, the equation of the line is $\frac{x}{2} + \frac{y}{3} = 1 \Rightarrow 3x + 2y = 6$.

11. Since intercepts are given so we can use the intercept form i.e. $\frac{x}{a} + \frac{y}{b} = 1$. Given that $a = -5$ and $b = 6$.

    Therefore, the equation of the line is $\frac{x}{-5} + \frac{y}{6} = 1 \Rightarrow 5y = 6x + 30$.

12. Let the intercept be $a$ on both the axes. Then the equation of the line would be $\frac{x}{a} + \frac{y}{a} = 1 \Rightarrow x + y = a$. Since the line passes through $(5, 6)$, therefore, $a = 5 + 6 = 11$. And thus, the equation of the straight line is $x + y = 11$.

    In the second case let the intercepts be $a, -a$. Then the equation would be $\frac{x}{a} - \frac{y}{a} = 1 \Rightarrow x - y = a$. Since the line passes through $(5, 6)$, therefore, $a = 5 - 6 = -1$. And thus, equation of the straight line would be $x - y + 1 = 0$.

13. First let the intercepts be $a, a$, then the equation of the line would be $\frac{x}{a} + \frac{y}{a} = 1 \Rightarrow x + y = a$. Since the line passes theough $(1, -2)$, therefore, $a = 1 - 2 = -1$. And thus, the equation fo the straight line is $x + y = -1$ or $x + y + 1 = 0$.

    Now let the intercepts be $a, -a$, then $a = x - y = 1 + 2 = 3$. So, the equation would be $x - y = 3$.

14. Let $a, b$ are the intercepts with $x$-axis and $y$-axis respectively. Since $(x', y')$ bisects it therefore $x' = \frac{a}{2} \Rightarrow a = 2x'$, and similarly, $b = 2y'$. The equation of line would be $\frac{x}{a} + \frac{y}{b} = 1$

    And thus, the required equation of the line in question is $xy' + x'y = 2x'y'$.

15. Let $a, b$ are the intercepts with $x$-axis and $y$-axis respectively. Since $(-4, 3)$ divides the intercept in the ratio $5 : 3$, therefore, $-4 = \frac{5.0+3.a}{5+3} \Rightarrow a = -\frac{32}{3}$ and $3 = \frac{5.b+3.0}{5+3} \Rightarrow b = \frac{24}{5}$.

    And thus equation of line is $\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow -\frac{3x}{32} + \frac{5y}{24} = 1 \Rightarrow 20y - 9x = 96$.

16. We will make use of two point form. The equation of the line is given by $y - 0 = \frac{-2-0}{2-0}(x - 0) \Rightarrow x + y = 0$.

17. We will make use of two point form. The equation of the line is given by $y - 4 = \frac{6-4}{5-3}(x - 3) \Rightarrow y = x + 1$.

18. We will make use of two point form. The equation of the line is given by $y - 3 = \frac{-7-3}{6+1}(x + 1) \Rightarrow 7y - 21 = -10x - 10 \Rightarrow 10x + 7y = 11$.

19. This problem can be solved with intercept form. Intercept on $x$-axis is $b$ and on $y$-axis is $-a$. Thus, equation of the line is $\frac{x}{b} - \frac{y}{a} = 1 \Rightarrow ax - by = ab$.

20. We will make use of two point form. The equation of the line is given by $y - b = \frac{a-b-b}{a+b-a}(x - a) \Rightarrow by - b^2 = (a - 2b)x - a^2 + 2ab \Rightarrow (a - 2b)x - by + b^2 + 2ab - a^2 = 0$.

21. The equation of the given line is given by $y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2) = \frac{2}{t_2 + t_1}(x - at_1^2)$

$\Rightarrow y(t_1 + t_2) - 2x = 2at_1t_2.$

22. The equation of the given line is given by $y - \frac{a}{t_1} = \frac{\frac{a}{t_2} - \frac{a}{t_1}}{at_2 - at_1}(x - at_1) = -\frac{1}{t_1t_2}(x - at_1)$

$\Rightarrow t_1t_2y + x = a(t_1 + t_2).$

23. The equation of the line is given by $y - a\sin\varphi_1 = \frac{a\sin\varphi_2 - a\sin\varphi_1}{a\cos\varphi_2 - a\cos\varphi_1}(x - a\cos\varphi_1)$

$\Rightarrow y - a\sin\varphi_1 = \frac{2\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_2-\varphi_1}{2}}{2\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}}(x - a\cos\varphi_1)$

$\Rightarrow x\cos\frac{\varphi_1+\varphi_2}{2} + y\sin\frac{\varphi_1+\varphi_2}{2} = a\cos\frac{\varphi_1-\varphi_2}{2}.$

24. The equation of the line is given by $y - b\sin\varphi_1 = \frac{b\sin\varphi_2 - b\sin\varphi_1}{a\cos\varphi_2 - a\cos\varphi_1}(x - a\cos\varphi_1)$

$\Rightarrow y - b\sin\varphi_1 = \frac{2b\cos\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_2-\varphi_1}{2}}{2a\sin\frac{\varphi_1+\varphi_2}{2}\sin\frac{\varphi_1-\varphi_2}{2}}(x - a\cos\varphi_1)$

$\Rightarrow \frac{x}{a}\cos\frac{\varphi_1+\varphi_2}{2} + \frac{y}{b}\sin\frac{\varphi_1+\varphi_2}{2} = \cos\frac{\varphi_1-\varphi_2}{2}$

25. The equation of the line is given by $y - b\tan\varphi_1 = \frac{b\tan\varphi_2 - b\tan\varphi_1}{a\sec\varphi_2 - a\sec\varphi_1}(x - a\sec\varphi_1)$

Now $\frac{b\tan\varphi_2 - b\tan\varphi_1}{a\sec\varphi_2 - a\sec\varphi_1} = \frac{b}{a}.\frac{\sin\varphi_2\cos\varphi_1 - \sin\varphi_1\cos\varphi_2}{\cos\varphi_1 - \cos\varphi_2}$

$= \frac{\sin(\varphi_2-\varphi_1)}{\cos\varphi_1 - \cos\varphi_2} = \frac{2\sin\left(\frac{\varphi_2-\varphi_1}{2}\right)\cos\left(\frac{\varphi_2-\varphi_1}{2}\right)}{2\frac{\sin(\varphi_1+\varphi_2)}{2}\frac{\sin(\varphi_2-\varphi_1)}{2}}$

Simplifying gives us the equation of the line as
$bx\frac{\cos(\varphi_1-\varphi_2)}{2} - ay\frac{\sin(\varphi_1+\varphi_2)}{2} = ab\frac{\cos(\varphi_1+\varphi_2)}{2}.$

26. Let the vertices of the triangle be $A(1,4)$, $B(2,-3)$ and $C(-1,-2)$.

We find the equations of the sides $AB$, $BC$ and $CA$.

Slope of $AB$: $m_{AB} = \frac{-3-4}{2-1} = -7$

Equation of $AB$: $y - 4 = -7(x - 1)$ so $y = -7x + 11$

Slope of $BC$: $m_{BC} = \frac{-2-(-3)}{-1-2} = \frac{1}{-3} = -\frac{1}{3}$

Equation of $BC$: $y + 3 = \left(-\frac{1}{3}\right)(x - 2)$ so $y = -\frac{1}{3}x - \frac{7}{3} \Rightarrow 3y + x + 7 = 0$

Slope of $CA$: $m_{CA} = \frac{4-(-2)}{1-(-1)} = \frac{6}{2} = 3$

Equation of $CA$: $y - 4 = 3(x - 1)$ so $y = 3x + 1$

Hence, the equations of the sides are: $AB : y + 7x = 11$, $BC : 3y + x + 7 = 0$, and $CA : y = 3x + 1$

27. Let the vertices of the triangle be $A(0,1)$, $B(2,0)$ and $C(-1,-2)$.

We find the equations of the sides $AB$, $BC$ and $CA$.

Slope of $AB$: $m_{AB} = \frac{0-1}{2-0} = -\frac{1}{2}$

Equation of $AB$: $y-1=\left(-\frac{1}{2}\right)(x-0)$ so $y=-\frac{1}{2}x+1$

Slope of $BC$: $m_{BC}=\frac{-2-0}{-1-2}=-\frac{2}{-}3=\frac{2}{3}$

Equation of $BC$: $y-0=\left(\frac{2}{3}\right)(x-2)$ so $y=\frac{2}{3}x-\frac{4}{3}$

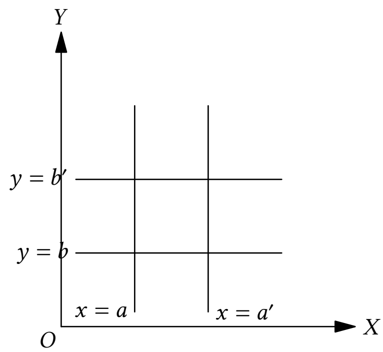
Slope of $CA$: $m_{CA}=\frac{1-(-2)}{0-(-1)}=\frac{3}{1}=3$

Equation of $CA$: $y-1=3(x-0)$ so $y=3x+1$

Hence, the equations of the sides are: $AB: 2y+x=2,\ BC: y=\frac{2}{3}x-\frac{4}{3}$, and $CA: y=3x+1$.

28. Intersection of $x=a, y=b$ will give the point $(a,b)$ and opposite to it will be intersection of the lines $x=a'$ and $y=b'$ i.e. $(a',b')$. Equation of this diagonal would be $y-b=\frac{b'-b}{a'-a}(x-a)\Rightarrow(b'-b)x-(a'-a)y+(a'-a)b+(b-b')a=(b'-b)x+(a-a')y+a'b-ab'$.

    Intersection of $x=a, y=b'$ will give the point $(a,b')$ and opposite to it will be intersection of the lines $x=a'$ and $y=b$ i.e. $(a',b)$. Equation of this diagonal would be $y-b=\frac{b-b'}{a'-a}(x-a')$, simplification of which is left to you.




29. Point which bisects the distance between $(a,b)$ and $(a',b')$ is given by $\left(\frac{a+a'}{2},\frac{b+b'}{2}\right)$, and point which bisects the distance between $(-a,b)$ and $(a',-b')$ is given by $\left(\frac{a'-a}{2},\frac{b-b'}{2}\right)$.

    The equation of the line passing through these points obtained is given by $y-\frac{b+b'}{2}=\frac{\frac{b-b'}{2}-\frac{b+b'}{2}}{\frac{a'-a}{2}-\frac{a+a'}{2}}\left(x-\frac{a+a'}{2}\right)$

    $\Rightarrow 2ay-2b'x=ab-a'b'$.

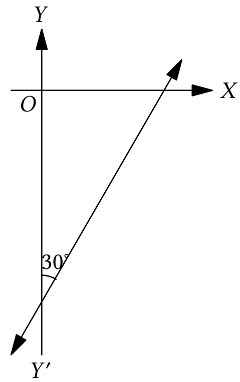
30. Intercepts of the line $3x+y=12$ are $(4,0)$ and $(0,12)$. The points which trisect these lines are $\left(\frac{2.4+0.1}{3},\frac{0.2+12.1}{3}\right)$ and $\left(\frac{1.4+0.2}{3},\frac{0.1+12.2}{3}\right)$ i.e. $\left(\frac{8}{3},4\right)$ and $\left(\frac{4}{3},8\right)$.

    Line passing through origin and $\left(\frac{8}{3},4\right)$ is $y=\frac{4}{\frac{8}{3}}x\Rightarrow 3x=2y$, and line passing through origin and $\left(\frac{4}{3},8\right)$ is $y=\frac{8}{\frac{4}{3}}x\Rightarrow y=6x$.

31. Slope of the line $=m=\tan 15^{\circ}=\tan(45^{\circ}-30^{\circ})=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=2-\sqrt{3}$

Intercept on $y$-axis is $c = -4$. Therefore, the equation of the line is $y = \left(2 - \sqrt{3}\right)x - 4$.

32.




From the diagram it is clear that angle made with positive direction of $x$-axis is $60^{\circ}$. Thus, slope of the line is $m = \tan 60^{\circ} = \sqrt{3}$, and the intercept with $y$-axis is $-4\sqrt{3}$.

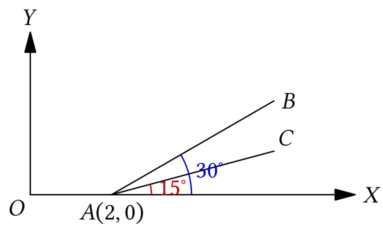
Thus, equation of the line is $\sqrt{3}x - y - 4\sqrt{3} = 0$.

33. Given that $\cos\theta = -\frac{1}{3} \Rightarrow \tan\theta = -\sqrt{3} = m$, which is slope of the line. Thus, the equation of the line is given by

$y - 2 = -\sqrt{8}(x - 1) \Rightarrow 2\sqrt{2}x + y - 2\left(\sqrt{2} + 1\right) = 0.$

34. Equation of the line is given by $y - 3 = \frac{3+2}{-1-4}(x + 1) \Rightarrow x + y = 2$.

35.




Given $A \equiv (2, 0)$, and $AB$ is the initial position, and $AC$ is the final position after rotation.

Given $\angle BAX = 30^{\circ}$, and $\angle BAC = 15^{\circ} \therefore \angle CAX = 15^{\circ}$

Slope of line $AC = \tan 15^{\circ} = 2 - \sqrt{3}$

Therefore, equation of line $AC$ is $y - 0 = \left(2 - \sqrt{3}\right)(x - 2) \Rightarrow \left(2 - \sqrt{3}\right)x - y - 4 + 2\sqrt{3} = 0$.

36. Let $AD$ be the internal bisector of the $\angle BAC$ which meets the side $BC$ at $D$.

Now $AB = \sqrt{(5-2)^2 + (2-3)^2} = \sqrt{10}$, and $AC = \sqrt{(5-6)^2 + (2-5)^2} = \sqrt{10}$

Since $AD$ is the internal bisector, therefore, $\frac{BD}{DC} = \frac{AB}{AC} = \frac{\sqrt{10}}{\sqrt{10}} = 1$

$\therefore D \equiv \left(\frac{2+6}{2}, \frac{3+5}{2}\right) = (4,4)$

Now equation of $AD$ is $y - 2 = \frac{2-4}{5-4}(x-5) \Rightarrow 2x + y = 12$.

37. Let $ABCD$ be a rectangle such that $A \equiv (1,2)$ and $C \equiv (5,5)$. Clearly, vertices $B$ and $D$ lie on the line $x = 3$. Let them be $B(3, y_1)$ and $D(3, y_2)$.

Since $AC$ and $BD$ bisect each other, therefore, their middle-points will be same. Thus, $\frac{y_1+y_2}{2} = \frac{2+5}{2} \Rightarrow y_1 + y_2 = 7$.

Also, $BD^2 = AC^2 \Rightarrow (y_1 - y_2)^2 = (1-5)^2 + (2-5)^2 = 25 \Rightarrow y_1 - y_2 = \pm 5$

$\Rightarrow y_1 = 6, y_2 = 1$ or $y_1 = 1, y_2 = 6$. So the other vertices are $(3,1)$ and $(3,6)$. Let $B$ represent $(3,1)$ and $D$ represent $(3,6)$.

Equation of side $AB$ is $y - 2 = \frac{2-1}{1-3}(x-1) \Rightarrow x + 2y = 5$.

Equation of side $BC$ is $y - 1 = \frac{1-5}{3-5}(x-3) \Rightarrow 2x - y = 5$.

Equation of side $CD$ is $y - 5 = \frac{5-6}{5-3}(x-5) \Rightarrow x + 2y = 15$.

Equation of side $AD$ is $y - 2 = \frac{2-6}{1-3}(x-1) \Rightarrow 2x = y$.

38. Equation of $OT$ : Slope of $OT = \tan 45^\circ = 1$ and it passes through $O(0,0)$.

Thus, equation is $y - 0 = 1.(x-0) \Rightarrow y = x$.

Equation of $OS$ : Slope of $OS = \tan 135^\circ = -1$ nad it passes through $O(0,0)$.

Thus, equation is $y - 0 = -1(x-0) \Rightarrow x + y = 0$.

Equation of $SP$ : Given $OT = 2\sqrt{2} \therefore OP = OT \sec 45^\circ = 4 \therefore P \equiv (0,4)$.

Also, slope of the line $SP$ is $\tan 45^\circ = 1$.

Thus, equation is $y - 4 = 1(x-0) \Rightarrow y = x + 4$.

Equation of $QR$ : Given $OQ = OT \sec 45^\circ = 4 \therefore Q \equiv (4,0)$.

Slope of line $QR = \tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = 2 + \sqrt{3}$.

Thus, equation is $y - 0 = \left(2+\sqrt{3}\right)(x-4) \Rightarrow \left(2+\sqrt{3}\right)x - y - 8 - 4\sqrt{3} = 0$.

Equation of $PR$ : $P \equiv (4,0)$. Slope of line $PR = \tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = 2 - \sqrt{3}$.

Thus, equation is $y - 4 = \left(2-\sqrt{3}\right)(x-0) \Rightarrow \left(2-\sqrt{3}\right)x - y + 4 = 0$.

Equation of $PQ$ : $P \equiv (0,4)$ and $Q \equiv = (4,0)$.

Thus, equation is $y - 4 = \frac{4-0}{0-4}(x-0) \Rightarrow x + y = 4$.

39. Let $AD, BE$ and $CF$ meet at $O$. We take $O$ as origin. Let the coordinates of points $A, B$ and $C$ be $(x_1, y_1), (x_2, y_2)$ and $(x_3, y_3)$ respectively.

Let $D$ divide $BC$ in the ratio $k:1$ i.e. $\frac{BD}{DC} = k$ then $D \equiv \left(\frac{kx_3+x_2}{k+1}, \frac{ky_3+y_2}{k+1}\right)$

Also, equation of line $AD$ is $y - 0 = \frac{y_1-0}{x_1-0}(x-0) \Rightarrow y = \frac{y_1}{x_1}x$

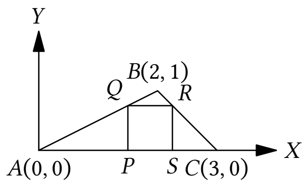
Since $D$ lies on $AD$, therefore, $\frac{ky_3+y_2}{k+1} = \frac{y_1}{x_1}\left(\frac{kx_3+x_2}{k+1}\right)$

$k = \frac{BD}{DC} = \frac{x_2y_1-x_1y_2}{x_1y_3-x_3y_1}$

Similarly, $\frac{CE}{EA} = \frac{x_3y_2-x_2y_3}{x_2y_1-x_1y_2}$, and $\frac{AF}{FB} = \frac{x_1y_3-x_3y_1}{x_3y_2-x_2y_3}$

Thus, $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$.

40.




Let $PQRS$ be the square inscribed in the ABC. Let $P \equiv (a, 0)$ and length of each side of the square be $k$ then $S \equiv (a+k, 0), Q \equiv (a, k), R \equiv (a+k, k)$.

Equation of the line $AB$ is $y - 0 = \frac{1-0}{2-0}(x-0) \Rightarrow x = 2y$.

Equation of the line $BC$ is $y - 0 = \frac{0-1}{3-2}(x-3) \Rightarrow x + y = 3$.

Since $Q(a, k)$ lies on $AB$, therefore, $a = 2k$.

Again $R(a+k, k)$ lies on $BC$, therefore, $a + 2k = 3 \Rightarrow k = \frac{3}{4}, a = \frac{3}{2}$.

Hence, $P \equiv \left(\frac{3}{2}, 0\right), Q \equiv \left(\frac{3}{2}, \frac{3}{4}\right), R \equiv \left(\frac{9}{4}, \frac{3}{4}\right)$ and $S \equiv \left(\frac{9}{4}, 0\right)$.

41. Equation of the given line is $\sqrt{3}y - 3x = 3 \Rightarrow y = \sqrt{3}x + \sqrt{3}$, which is of the form $y = mx + c$.

Slope of the line is $\sqrt{3} = \tan 60^\circ$. Thus, the given line makes an angle of $60^\circ$ with the $x$-axis.

42. Since slope and intercept are given, therefore, slope-intercept form can be used. Given that $m = 3, c = 7$, therefore, equation of the straight line is $y = 3x + 7$.

43. Since slope and intercept are given, therefore, slope-intercept form can be used. Given that

$m = \tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = 2 + \sqrt{3}$, and $c = 3$.

Therefore, the equation of the line is $y = \left(2 - \sqrt{3}\right)x + 3$.

44. Since slope and intercept are given, therefore, slope-intercept form can be used. Given that

$m = \tan\sin^{-1}\frac{12}{13} = \frac{5}{13}$ and $c = -5$

Therefore, the equation of the line is $y = \frac{5}{13}x - 5 \Rightarrow 5x - 13y = 65$.

45. Since the line is parallel to $x$-axis, therefore, it will make an angle of $0^\circ$ with $x$ -axis i.e. $m = \tan 0^\circ = 0$. Also, since its distance from $x$-axis is 5units, therefore, the intercept on $y$-axis is 5, which makes intercept $c = 5$.

    Thus, equation of the line would be $y = 0.x + 5 \Rightarrow y = 5$.

    Since it is not given that intercept is from positive or negative direction of $y$-axis, therefore, the other line would be $y + 5 = 0$.

46. Since the line is parallel to $y$-axis therefore the equation would be $x = k$, where $k$ is the intercept on $x$-axis, which is given as $-4$. Therefore, the equation of the line is $x = -4$.

47. Lines parallel and perpendicular to $x$-axis are given by $x = k$ and $y = p$, where $k$ and $p$ are distance of the line from the $y$-axis and $x$-axis.

    Since these lines pass through $(5, 3)$, therefore, $x = 5$ and $y = 3$ are the desired equations of the straight lines.

48. Since the line makes an angle of $135^\circ$ with positive direction of the $y$-axis, therefore, it makes an angle of $135^\circ$ with positive direction of the $x$-axis. Thus, slope of the line is $m = \tan 135^\circ = -1$.

    Also give that it cuts an intercept of 2 from positive direction of the $x$-axis, which means that it passes through $(2, 0)$.

    Thus, equation of the straight line would be $y - 0 = -1.(x - 2) \Rightarrow x + y = 2$.

49. Since the slope is 2 and the line cuts an intercept of 4 on $x$-axis i.e. it passes through $(4, 0)$ the equation of the line would be

    $y = 0 = 2(x - 4) \Rightarrow 2x - y = 8$.

50. Since the line makes an angle of $60^\circ$ with the positive direction of the $y$-axis, therefore, it would make an angle of $30^\circ$ with the positive direction of $x$-axis. Therefore, the slope of the line is $m = \tan 30^\circ = \frac{1}{\sqrt{3}}$.

    Also given that the line passes through $(3, -2)$, thus the equation of the line would be

    $y + 2 = \frac{1}{\sqrt{3}}(x - 3) \Rightarrow x - \sqrt{3}y = 3 + 2\sqrt{3}$.

51. Slope is given by $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-4}{1-3} = 1$.

    The equation of the line would be $y - 4 = 1.(x - 3) \Rightarrow x - y + 1 = 0$.

52. The equation of the line is given by $y - b = \frac{b + r\sin\theta - b}{a + r\cos\theta - a}(x - a)$

    $\Rightarrow x\tan\theta - y = a\tan\theta - b$.

53. The equation of the straight line is given by $y + 3 = \frac{2+3}{-4-1}(x - 1) \Rightarrow x + y + 2 = 0$.

54. Equation of the straight line passing through $(1, 4)$ and $(3, -2)$ is given by

$y - 4 = \frac{-2-4}{3-1}(x-1) \Rightarrow 3x + y = 7.$

Now we put $(-3, 16)$ in this equation which gives us $3.-3+16=7$, which is true. Thus, the point $(-3, 16)$ also lies on the same line making the points collinear. We could have found the equation between $(3, -2)$ and $(-3, 16)$ which would also give the same equation.

Another way would be finding the area of the triangle whose vertices are the given three points and we will find that area of the triangle is zero; making the points collinear.

55. Line passing through $(a, b)$ and $(a_1, b_1)$ is given by

$y - b = \frac{b_1-b}{a_1-a}(x-a) \Rightarrow (b_1-b)x - a(b_1-b) = (a_1-a)y - b(a_1-a)$

$\Rightarrow (b_1-b)x - ab_1 = (a_1-a)y - a_1b$

Now $(a-a_1, b-b_1)$ also lies on this point, therefore, it should satisfy the above equation.

Thus, $(b_1-b)(a-a_1) - ab_1 = (a_1-a)(b-b_1) - a_1b \Rightarrow ab_1 - ab - a_1b_1 + a_1b - ab_1 = a_1b - a_1b_1 - ab + ab_1 - a_1b$

$\Rightarrow ab_1 = a_1b.$

Thus, the equation of the line becomes $(b_1-b)x = (a_1-a)y$, which clearly passes through the origin.

56. The equation of the straight line which passes through $(1, 2)$ and $(-3, 0)$ is given by

$y - 2 = \frac{0-2}{-3-1}(x-1) \Rightarrow 2y = x + 3.$

For the points to be collinear $(t-1, 3)$ has to be on this line. Thus,

$2.3 = t - 1 + 3 \Rightarrow t = 4.$

57. The equation of the straight line which passes through $(p, q+r)$ and $(q, r+p)$ is given by

$y - q - r = \frac{r+p-q-r}{q-p}(x-p) \Rightarrow x + y = p + q + r.$

If the line passes through $(r, p+q)$ then it would satisfy the obtained equation of the line. Putting the point in the obtained equation we have

$r + p + q = p + q + r$, which is true. Hence, proved.

58. Point dividing the line segment joining the points $(-1, 2)$ and $(4, -5)$ externally in the ratio $2:3$ is given by

$\left(\frac{2.4-3.-1}{2-3}, \frac{2.-5-3.2}{2-3}\right) \equiv (-11, 16).$

The equation of the line passing through $(1, 2)$ and $(-11, 16)$ is given by

$y - 2 = \frac{16-2}{-11-1}(x-1) \Rightarrow 7x + 6y - 19 = 0.$

59. The equation of $BC$ is given by $y - 1 = \frac{0-1}{2-0}(x-0) \Rightarrow x + 2y = 2.$

The vertex $A$ is $(-1,-2)$ and median passing through it will bisect $BC$ i.e. it will pass through the point $\left(1,\frac{1}{2}\right)$.

Thus, equation of the median is given by

$y+2=\frac{\frac{1}{2}+2}{1+1}(x+1) \Rightarrow 5x-4y-3=0.$

60. The mid-point of $(2,3)$ and $(5,4)$ is $\left(\frac{7}{2},\frac{7}{2}\right)$. The equation of the median passing through $(1,2)$ and $\left(\frac{7}{2},\frac{7}{2}\right)$ is given by

$y-2=\left(\frac{7}{2}-2\right)\left(\frac{7}{2}-1\right)(x-1) \Rightarrow 3x-5y+7=0.$

The mid-point of $(1,2)$ and $(2,3)$ is $\left(\frac{3}{2},\frac{5}{2}\right)$. The equation of the median passing through $(5,4)$ and $\left(\frac{3}{2},\frac{5}{2}\right)$ is given by

$y-4=\frac{\frac{5}{2}-4}{\frac{3}{2}-5}(x-5) \Rightarrow 3x-7y+13=0.$

The mid-point of $(1,2)$ and $(5,4)$ is $(3,3)$. The equation of the median passing through $(2,3)$ and $(3,3)$ is given by

$y-3=\frac{3-3}{3-2}(x-2) \Rightarrow y=3.$

61. Let the line segment joining $A(2,3)$ and $B(-1,4)$ be divided by the line $x+y+1=0$ in the ratio $m:n$.

Using the section formula, the point of division $P$ is $P=\left(\frac{m*(-1)+n*2}{m+n},\frac{m*4+n*3}{m+n}\right)$.

Since $P$ lies on $x+y+1=0$, substitute: $\left(\frac{-m+2n}{m+n}\right)+\left(\frac{4m+3n}{m+n}\right)+1=0$

$\Rightarrow \frac{-m+2n+4m+3n}{m+n}+1=0 \Rightarrow \frac{3m+5n}{m+n}+1=0 \Rightarrow \frac{m}{n}=-\frac{3}{2}$

Hence, the line divides the segment externally in the ratio $-3:2$, i.e., $3:2$ externally.

62. Let $A(2,3)$ and $B(4,1)$. Let the line through $(1,2)$ and $(4,3)$ divide $AB$ in the ratio $m:n$ at point $P$.

Point $P=\left(\frac{m.4+n.2}{m+n},\frac{m.1+n.3}{m+n}\right)$.

Slope of line through $(1,2)$ and $(4,3)$ is $\frac{3-2}{4-1}=\frac{1}{3}$.

Equation of this line: $y-2=\left(\frac{1}{3}\right)(x-1)$.

Substituting $P$: $\left(\frac{m+3n}{m+n}\right)-2=\left(\frac{1}{3}\right)\left(\frac{4m+2n}{m+n}-1\right)$

Simplifying LHS: $\frac{m+3n-2m-2n}{m+n}=\frac{-m+n}{m+n}$

RHS: $\left(\frac{1}{3}\right)\left(\frac{4m+2n-m-n}{m+n}\right)=\left(\frac{1}{3}\right)\left(\frac{3m+n}{m+n}\right)$

So, $\frac{-m+n}{m+n}=\frac{3m+n}{3(m+n)} \Rightarrow n=3m \Rightarrow m:n=1:3.$

Hence, the line divides the segment internally in the ratio $1:3$.

63. $D\equiv\left(\frac{2.1+1.-1}{2+1},\frac{2.-3+1.-2}{2+1}\right)=\left(\frac{1}{3},-\frac{8}{3}\right)$

Let mid-point of $AC$ is $M$ then $M\equiv\left(\frac{3}{2},-\frac{1}{2}\right)$.

Equation of $BM$ is given by $y+2=\frac{-\frac{1}{2}+2}{\frac{3}{2}+1}(x+1) \Rightarrow 5y+10=3x+3 \Rightarrow 3x-5y=7$

Equation of $AD$ is given by $y-2=\frac{-\frac{8}{3}-2}{\frac{1}{3}-2}(x-2) \Rightarrow 5y-10=14x-28 \Rightarrow 14x-5y=18$

The point of intersection of two obtained equations is given by $\left(1,-\frac{4}{5}\right)$.

Let this point divide $BM$ in the ratio of $k:1$, then

$1=\left(k*\frac{3}{2}+1.-1\right)(k+1) \Rightarrow k+1=\frac{3}{2}k-1 \Rightarrow k=4$. Thus ratio is $4:1$.

64. The equation of the the line can be written as $y=\sqrt{3}x+3$. Comparing it will $y=mx+c$ gives us $m=\sqrt{3}$ and $c=3$.

    Thus, slope of the line $m=\sqrt{3}=\tan 60^{\circ}$. Thus, the line makes an angle of $60^{\circ}$ with the positive direction of the $x$-axis.

    $c=3$ tells us that the intercept on $y$-axis is 3 in positive direction.

65. Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$.

    It is given that $b=2a$ which makes the equation of the line $2x+y=2a$.

    Since it passes through $(3,4)$, therefore, $2.3+4=2a \Rightarrow a=5$, which makes the equation $2x+y=10$.

66. Let the equation of the line is $\frac{x}{a}+\frac{y}{b}=1$ so the point on $x$-axis where this line meets is $(a,0)$ and on $y$-axis it is $(0,b)$.

    Given that $(3,4)$ divdes the line segment joining $(a,0)$ and $(b,0)$ in the ratio of $2:3$, therefore,

    $3=(2.0+3.a)(2+3) \Rightarrow a=5$ and $4=\frac{2.b+3.0}{2+3} \Rightarrow b=10$

    Thus, equation of the line is $2x+y=10$.

67. The line $3x+4y=12$ can be written as $\frac{x}{4}+\frac{y}{3}=1$ so the intercept of $x$-axis is 4 and the intercept on $y$-axis is 3.

    Thus, according to the question the required line makes an intercept of 8 on $x$-axis and 9 on $y$-axis. Thus, the required line is

    $\frac{x}{8}+\frac{y}{9}=1 \Rightarrow 9x+8y=72$.

68. $ax+by+c=0$ can be written as $-\frac{x}{\frac{c}{a}}-\frac{y}{\frac{c}{b}}=1$. Thus, intercept on $x$-axis is $-\frac{c}{a}$ and on $y$-axis is $-\frac{c}{b}$.

    Let the equation of the line be $y=mx+c$, but since the line passes through origin $c=0$.

    Now mid-point of the intercept is given by $\left(-\frac{c}{2a},-\frac{c}{2b}\right)$. Putting this point in the line

    $-\frac{c}{2b}=-m\frac{c}{2a} \Rightarrow m=\frac{a}{b}$, which makes the line $ax=by$.

69. Given line is $3x + 4y = 12 \Rightarrow \frac{x}{4} + \frac{y}{3} = 1$. Let this line cut $x$ and $y$ axes at $A$ and $B$ respectively. Then $A \equiv (4, 0)$ and $B \equiv (0, 3)$.

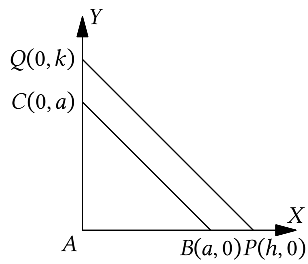
    Let $P$ and $Q$ be the points which trisect $AB$ such that $\frac{AP}{PB} = 1 : 2$ and $\frac{AQ}{BQ} = 2 : 1$

    $\Rightarrow P \equiv \left(\frac{1.0+2.4}{3}, \frac{1.3+2.0}{3}\right) = \left(\frac{8}{3}, 1\right)$ and $Q \equiv \left(\frac{2.0+1.4}{3}, \frac{2.3+1.0}{3}\right) = \left(\frac{4}{3}, 2\right)$

    Equation of the line passing through origin and $P$ is given by $y - 0 = \frac{1-0}{\frac{8}{3}-0}(x - 0) \Rightarrow 3x - 8y = 0$.

    Equation of the line passing through origin and $Q$ is given by $y - 0 = (2 - 0)\left(\frac{4}{3} - 0\right)(x - 0) \Rightarrow 3x - 2y = 0$.

70. Let the line be $\frac{x}{a} + \frac{y}{b} = 1$, which will cut intercepts $a$ and $b$. According to question $\frac{1}{a} + \frac{1}{b} = k$, where $k$ is a constant. Thus, $\frac{\frac{1}{k}}{a} + \frac{\frac{1}{k}}{b} = 1$, which passes through the point $\left(\frac{1}{k}, \frac{1}{k}\right)$.

71. 


    Let $ABC$ be a right angles isosceles triangle in which $AB = AC$. We take $A$ as the origin and $AB$ and $AC$ as $x$ and $y$ axes respectively. Let $AB = AC = a$.

    Also, let $AP = h, AQ = k$. The equation of the line $PQ$ is $\frac{x}{h} + \frac{y}{k} = 1$

    Given that $BP.CQ = AB^2 \Rightarrow (h - a)(k - a) = a^2 \Rightarrow \frac{a}{h} + \frac{a}{k} = 1$, which shows that $PQ$ passes through the point $(a, a)$.

72. Given that $P \equiv (\alpha, \beta)$ and the equation of the line $\frac{x}{a} + \frac{y}{b} = 1$.

    The line will cut the axes at $(a, 0)$ and $(0, b)$. Given that $\Delta OAB = \frac{1}{2}|ab| = S \Rightarrow 2S = ab$, where $O$ is the origin.

    Since the line passes through $P$, therefore $\frac{\alpha}{a} + \frac{\beta}{b} = 1 \Rightarrow \frac{\alpha}{a} + \frac{a\beta}{2}S = 1$

    $\Rightarrow a^2\beta - 2aS + 2\alpha S = 0$, which is a quadratic equation in $a$. However, $a$ is real, therefore $D = 4S^2 - 8\alpha\beta Sg = 0 \Rightarrow Sg = 2\alpha\beta$

    Thus, the least value of $S$ is $2\alpha\beta$.

73. The equation of the line will be given by $x\cos 75^\circ + y\sin 75^\circ = 3\sqrt{2}$

    Now $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$ and $\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

    So the equation of the line is $\left(\sqrt{3} - 1\right)x + \left(\sqrt{3} + 1\right)y = 12$.

74. Slope is given as $\frac{5}{12}$ so if $\tan\theta = \frac{5}{12}$ then $\theta$ can lie in first or third quadrant. Thus, $\cos\theta = \pm\frac{12}{13}$ and $\sin\theta = \pm\frac{5}{13}$.

Equation of the line will be $x.\frac{12}{13} + y.\frac{5}{13} = 2 \Rightarrow 12x + 5y - 26 = 0$

or $x.\left(-\frac{12}{13}\right) + y.\left(-\frac{5}{13}\right) = 2 \Rightarrow 12x + 5y + 26 = 0$.

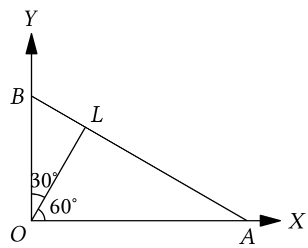
75. We can treat this place as origin, east direction as $x$-axis and north direction as $y$-axis. Then the angle made by perpendicular from the place to the line will be $45^{\circ}$ as the direction of the canal is north-east.

Thus, equation for this canal would be $\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{9}{2} \Rightarrow x + y = \frac{9}{\sqrt{2}}$.

The coordinate of the village is given by $(3, 4)$; putting this in the equation for the canal gives us

$3 + 4 = \frac{9}{\sqrt{2}}$, which is false. Hence, the village does not lie on the canal.

76.




Let the reuired line be $AB$ and $OL$ is perpendicular from the origin $O$ to $AB$. According to the question $OL$ makes an angle of $30^{\circ}$ with $y$-axis i.e. it will make an angle of $60^{\circ}$ with $x$-axis.

Let $OL = p$, so the equation of the line will be $x\cos 60^{\circ} + y\sin 60^{\circ} = p \Rightarrow x + \sqrt{3}y = 2p$.

Intercept on $x$-axis is $2p$ and intercept on $y$axis is $2\frac{p}{\sqrt{3}}$.

Also given that $\Delta OAB = 96\sqrt{3} = \frac{1}{2}OA.OB \Rightarrow 96\sqrt{3} = \frac{1}{2}2p.\frac{2p}{\sqrt{3}} \Rightarrow p = 12$

Thus, equation of the line is $x + \sqrt{3}y = 24$.

77. Given $OC = 2, \angle ABC = 90^{\circ}$ and $AB = BC$

$\therefore \angle BCA = \angle BAC = 45^{\circ}, OB = OC = 2$, and $BC = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

Let $OM$ be perpendicular to $DE$. $\because OC = 2, OB = 2$, therefore, equation of $BC$ will be

$\frac{x}{2} + \frac{y}{2} = 1 \Rightarrow x + y = 2$

Also, equation of $AB$ will be $\frac{x}{-}2 + \frac{y}{2} = 1 \Rightarrow x - y = -2$

$\because \angle LAC = 45^{\circ}$ and $OM = OL + LM = OC\cos 45^{\circ} + LM = OC.\frac{1}{\sqrt{2}} + BC = 3\sqrt{2}$

Thus, equation of $DE$ is $x\cos 45^{\circ} + y\sin 45^{\circ} = 3\sqrt{2} \Rightarrow x + y = 6$.

78. Given equation is $\sqrt{3}x + y = 8 \Rightarrow \frac{x}{\frac{8}{\sqrt{3}}} + \frac{y}{8} = 1$, which will meet $x$ and $y$ axes at $\left(\frac{8}{\sqrt{3}}, 0\right)$ and $(0, 8)$.

    The equation can be rewritten as $\frac{\sqrt{3}}{2}x + \frac{y}{2} = 4 \Rightarrow x\cos 30^\circ + y\sin 30^\circ = 4$, which is the equation in normal form.

    The length of perpendicular on this line from origin is 4 and it makes an angle of $30^\circ$ with the $x$-axis.

79. Let the equation of a line in intercept form be $\frac{x}{a} + \frac{y}{b} = 1$.

    Since it passes through $(3, 2)$, we have: $\frac{3}{a} + \frac{2}{b} = 1$.

    Given $a - b = 2$, so $a = b + 2$.

    Substitute into the first equation: $\frac{3}{b+2} + \frac{2}{b} = 1$.

    Multiplying by $b(b+2)$: $3b + 2(b+2) = b(b+2), b^2 - 3b - 4 = 0$

    Solving: $b = \frac{3 \pm 5}{2}$. So, $b = 4$ or $b = -1$.

    Then $a = b + 2$ gives: If $b = 4$, then $a = 6$. If $b = -1$, then $a = 1$.

    Thus, the required lines are: $\frac{x}{6} + \frac{y}{4} = 1$ and $x - y = 1$.

80. Let $A$ be $(a, 0)$ abd $B$ be $(0, b)$, then the equation of line will be given by $\frac{x}{a} + \frac{y}{b} = 1$.

    Since it passes through $P(1, -7)$, therefore, $\frac{1}{a} - \frac{7}{b} = 1 \Rightarrow \frac{1}{a} = \frac{7+b}{b} \Rightarrow a = \frac{b}{7+b}$

    Also given that $4AP = 4BP \Rightarrow 16[(a-1)^2 + 7^2] = 9[1^2 + (-7-b)^2]$

    $\Rightarrow 16(a-1)^2 + 784 = 9 + (b+7)^2$

    Putting the value of $a$ from above we get $b = -\frac{49}{3}$ and $a = \frac{7}{4}$

    Thus, equation of the line is $28x - 3y = 49$.

81. Let $A$ be $(a, 0)$ abd $B$ be $(0, b)$, then the equation of line will be given by $\frac{x}{a} + \frac{y}{b} = 1$.

    Since it passes through $P(2, 6)$, therefore, $\frac{2}{a} + \frac{6}{b} = 1 \Rightarrow \frac{2}{a} = \frac{b-6}{b} \Rightarrow a = 2\frac{b}{b-6}$

    Also given that $3AP = 2BP \Rightarrow 9[(a-2)^2 + 6^2] = 4[(-2)^2 + (b-6)^2]$

    Putting $a = 2\frac{b}{b-6}$ and solving gives us $a = \frac{10}{3}$ and $b = 15$

    Thus, equation of the line is $9x + 2y = 30$.

82. Given line is $3x + 4y = 6 \Rightarrow \frac{x}{2} + \frac{y}{\frac{3}{2}} = 1$. Thus, intercepts on axes are 2 and $\frac{3}{2}$ respectively.

    Double of these intercepts is 4 and 3. Thus, equation of line which makes these intercepts is

    $\frac{x}{4} + \frac{y}{3} = 1 \Rightarrow 3x + 4y = 12$.

83. Given line is $3x - 5y = 15 \Rightarrow \frac{x}{5} - \frac{y}{3} = 1$. Thus, points of interception are $(5, 0)$ and $(0, -3)$. Midpoint of intercepted portion will be $\left(\frac{5}{2}, -\frac{3}{2}\right)$.

The required line also passes through $(2, 1)$, hence in two-point form equation of the line will be

$y - 1 = \frac{-\frac{3}{2}-1}{\frac{5}{2}-2}(x - 2) \Rightarrow 5x + y = 11.$

84. The given line is $2x + 3y = 6 \Rightarrow \frac{x}{3} + \frac{y}{2} = 1$, thus points of interception are $(3, 0)$ and $(0, 2)$.

Let the points be $P(x_1, y_1)$ and $Q(x_2, y_2)$ which divide the intercepted points in the ratio of $2 : 1$ and $1 : 2$ respectively.

Thus, $P \equiv \left(\frac{2.0+1.3}{3}, \frac{2.2+1.0}{3}\right) \equiv \left(1, \frac{4}{3}\right)$ and $Q \equiv \left(\frac{1.0+2.3}{3}, \frac{1.2+2.0}{3}\right) \equiv \left(2, \frac{2}{3}\right)$.

Since these lines also pass through origin so the equations are given by $y = \frac{4}{3}x \Rightarrow 4x - 3y = 0$ and $y = \frac{\frac{2}{3}}{2}x \Rightarrow x - 3y = 0$.

85. Equation of the line in two-point form is given by $y - 1 = \frac{4-1}{11-5}(x - 5) \Rightarrow 2y - 2 = x - 5 \Rightarrow x - 2y - 3 = 0$

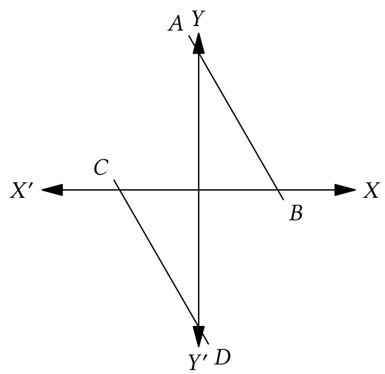
Putting $(1, -1)$ in the obtained equation for the line $1 - 2(-1) - 3 = 0$, which is true, so all points lie on the line $x - 2y = 3 \Rightarrow \frac{x}{3} + \frac{y}{-\frac{3}{2}} = 1$.

Thus, intercepts on the axes are $(3, 0)$ and $\left(0, -\frac{3}{2}\right)$ and intercepts between the axes is $\sqrt{3^2 + \left(\frac{3}{2}\right)^2} = \frac{3\sqrt{5}}{2}$.

86. Equation of the line in two-point form is given by $y + 3 = \frac{5+3}{4-1}(x - 1) \Rightarrow 3y + 9 = 8x - 8 \Rightarrow 8x - 3y = 17 \Rightarrow \frac{x}{\frac{17}{8}} + y\left(-\frac{17}{3}\right)$

Thus, intercepts on the axes are $\frac{17}{8}$ and $-\frac{17}{3}$ respectively.

87. There are two possibilities as shown in the diagram because length is a scalar quantity.



Since the line makes an angle of $150^\circ$ with positive direction of $y$-axis so it will make an angle of $120^\circ$ with positive direction of $x$-axis.

Thus, angle made by perpendicular with $x$-axis would be $30^\circ$ or $210^\circ$ with positive direction of $x$-axis.

Thus, equation of the line is $x\cos 30^\circ + y\sin 30^\circ = 7$ and $x\cos 210^\circ + y\sin 210^\circ = 7$

Thus, lines are given by $\sqrt{3}x + y = \pm 7$.

88. Since the perpendicular makes an angle with positive direction of $y$-axis with $30^\circ$ it will make an angle of $60^\circ$ with positive direction of $x$-axis. Also, given that length of the perpendicular from origin is 2. Therefore, the equation in normal form is given by

    $x\cos 60^\circ + y\sin 60^\circ = 2 \Rightarrow x + \sqrt{3}y = 4$.

89. The equation of the line in normal form is given by

    $x\cos 60^\circ + y\sin 60^\circ = 5 \Rightarrow x + \sqrt{3}y = 10$.

90. Given that $\tan\theta = \frac{3}{4}$ where $\theta$ is the angle made by the perpendicular with the positive direction of $x$-axis. Thus, $\tan\theta$ can also be $\frac{-3}{-4}$ i.e. in third quadrant.

    $\Rightarrow \cos\theta = \pm\frac{4}{5}, \sin\theta = \pm\frac{4}{5}$ and the equation in normal form will be

    $x\cos\theta + y\sin\theta = 6 \Rightarrow 4x + 3y = \pm 30$.

91. The equation of the line joining the points $(1, 2)$ and $(-3, 1)$ is given by

    $y - 2 = \frac{1-2}{-3-1}(x-1) \Rightarrow x - 4y + 7 = 0$, which can be written as $\frac{x}{7} + \frac{y}{-\frac{7}{4}} = 1$

    Thus, intercepts on axes are 7 and $-\frac{7}{4}$.

    $\cos\theta = \frac{7}{\sqrt{7^2 + \frac{7^2}{4^2}}} \Rightarrow p = \frac{7}{\sqrt{17}}$.

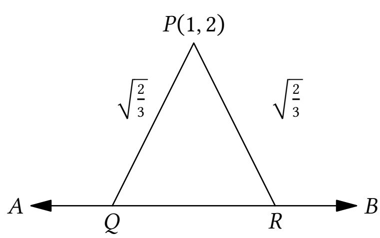
92. Let $P \equiv (3, 2)$ and let the required line make an angle $\theta$ with the positive direction of $x$-axis.

    Given $\tan\theta = \frac{3}{4}$.

    So the equation of the line is $y - 2 = \frac{3}{4}(x-3) \Rightarrow 3x - 4y - 1 = 0$.

    Coordinates of the points which are at a distance of 5 units from $P$ are $(3 \pm 5\cos\theta, 2 \pm 5\sin\theta) \Rightarrow (3 \pm 4, 2 \pm 3) \Rightarrow (7, 5)$ or $(-1, -1)$.

93. 



Let $P \equiv (1, 2)$. Let $AB$ be the given line $x + y = 4$.

Let the line through $P$ makes an angle $\theta$ with the $x$-axis cuts the line $AB$ at $Q$ and $R$ at a distance $\sqrt{\frac{2}{3}}$ from $P$. Then

$Q \equiv \left(1 + \sqrt{\frac{3}{2}} \cos\theta, 2 + \sqrt{\frac{2}{3}} \sin\theta\right)$

Since $Q$ lies on the line $AB$ therefore $1 + \sqrt{\frac{3}{2}} \cos\theta + 2 + \sqrt{\frac{2}{3}} \sin\theta = 4$

$\Rightarrow \cos\theta + \sin\theta = \sqrt{\frac{3}{2}} \Rightarrow \frac{1}{\sqrt{2}} \cos\theta + \frac{1}{\sqrt{2}} \sin\theta = \frac{\sqrt{3}}{2}$

$\Rightarrow \cos(\theta - 45^\circ) = \cos 30^\circ \Rightarrow \theta - 45^\circ = 2n\pi \pm 30^\circ$

$\Rightarrow \theta = 15^\circ, 75^\circ$.

94. Given line is $\sqrt{3}x - 4y + 8 = 0$ and $P \equiv \left(\sqrt{3}, 2\right)$. Let the line through $P$ making an angle of $\frac{\pi}{6}$ with the $x$-axis meet the libe at $Q$. Let $PQ = r$, then

$Q \equiv \left(\sqrt{3} + r\cos\frac{\pi}{6}, 2 + r\sin\frac{\pi}{6}\right) \equiv \left(\sqrt{3} + \frac{\sqrt{3}}{2}r, r + \frac{r}{2}\right)$

However, $Q$ lies on the given line, therefore,

$\sqrt{3}\left(\sqrt{3} + \frac{\sqrt{3}}{2}\right) - 4(r + \frac{r}{2}) + 8 = 0 \Rightarrow r = 6.$

95. Let $P \equiv (-2, 3)$. We know that the coordinates of points on the line making an angle $\theta$ with the positive direction of $x$-axis at a distance $r$ from a point $(x_1, y_1)$ are $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$.

Thus, required coordinates are $\left(-2 \pm 4\sqrt{2}\cos 45^\circ, 3 \pm 4\sqrt{2}\sin 45^\circ\right)$ i.e. $(2, 7)$ and $(-6, -1)$.

96. Given $A \equiv (2, 0)$ and $B \equiv (3, 1)$. Slope of the line $AB = \frac{0-1}{2-3} = 1 = \tan 45^\circ$.

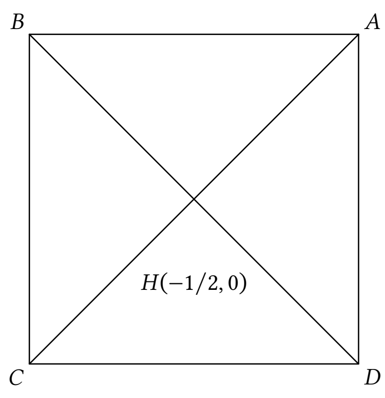
Thus, slope of the line $AC = 45^\circ + 15^\circ = 60^\circ$

Therefore, equation of the line $AC$ is $y - 0 = \tan 60^{\circ (x-2)} \Rightarrow \sqrt{3}x - y = 2\sqrt{3}$.

$AC = AB = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}$

Thus, $C \equiv \left(2 + \sqrt{2}\cos 60^\circ, 0 + \sqrt{2}\sin 60^\circ\right) = \left(2 + \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right)$.

97.



Let $A = (1,1)$ and $C = (-2,-1)$ then $H = \left(-\frac{1}{2}, 0\right)$, which is mid-point of $AC$ and $BD$.

Slope of $AC = \frac{1+1}{1+2} = \frac{2}{3} = \tan\theta$, therefore, slope of $BC = -\frac{3}{2}$ because diagonals of a square are perpendicular to each other.

Thus, $\theta$ is an obtuse angle. $\therefore \cos\theta = -\frac{2}{\sqrt{13}}$ and $\sin\theta = \frac{3}{\sqrt{13}}$

Also, $AC = \sqrt{13}$, therefore, $DH = \frac{\sqrt{13}}{2}$

Thus, coordinates of $B$ and $D$ are $\left(-\frac{1}{2} \pm \frac{\sqrt{13}}{2}\cos\theta, 0 \pm \frac{\sqrt{13}}{2}\sin\theta\right)$ i.e. $\left(-\frac{3}{2}, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

98. Let the line through $A$ making an angle $\theta$ with the positive direction at $x$-axis. Let $AB = r_1, AC = r_2$ and $AD = r_3$.

$B = (k + 1 + r_1\cos\theta, 2k + r_1\sin\theta)$. Since $B$ lies on $7x + y - 16 = 0$, therefore,

$7(k + 1 + r_1\cos\theta) + 2k + r_2\sin\theta - 16 = 0 \Rightarrow r_1 = \frac{9(1-k)}{7\cos\theta + \sin\theta}$

Also, $C = (k + 1 + r_2\cos\theta, 2k + r_2\sin\theta)$. Subce $C$ lies on the line $5x - y - 8 = 0$, therefore,

$5(k + 1 + r_2\cos\theta) - (2k + r_2\sin\theta) - 8 = 0 \Rightarrow r_2 = \frac{3(1-k)}{5\cos\theta - \sin\theta}$

Again $D = (k + 1 + r_3\cos\theta, 2k + r_3\sin\theta)$ and $D$ lies on the line $x - 5y + 8 = 0$, therefore,

$k + 1 + r_3\cos\theta - 5(2k + r_3\sin\theta) + 8 = 0 \Rightarrow r_r = \frac{9(1-k)}{5\sin\theta - \cos\theta}$

$\frac{1}{r_2} + \frac{1}{r_3} = \frac{2(7\cos\theta + \sin\theta)}{9(1-k)} = \frac{2}{r_1}$.

Hence, $r_2, r_1, r_3$ are in H.P.

99. Let $ABCD$ be the square whose center is $O$. Now $AO = \sqrt{5}$ and slope of $AO = \frac{1-0}{2-0} = \frac{1}{2} = \tan\theta$

$\therefore \cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$

$\therefore$ Coordinates of the points of $AC$ which are at a distance $\sqrt{5}$ from $O$ will be $\left(0 \pm \sqrt{5}\cos\theta, \pm\sqrt{5}\sin\theta\right) = (\pm 2, \pm 1)$

i.e. $(2,1)$ and $(-2,-1)$. Thus, $C \equiv (-2,-1)$

But $BD \perp AC$. So slope of $BD = -2 = \tan\alpha$(say)

$\therefore \frac{\pi}{2} < \alpha < \pi$ or $\frac{3\pi}{2} < \alpha < 2\pi$

$\therefore \cos\alpha = -\frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{5}}$ or $\cos\alpha = \frac{1}{\sqrt{5}}$ and $\sin\alpha = -\frac{2}{\sqrt{5}}$

Since $B$ and $D$ are on $BD$ at a distance $\sqrt{5}$ from $O$, their coordinates(in some order) will be

$\left(0 \pm \sqrt{5}\cos\alpha, 0 \pm \sqrt{5}\sin\alpha\right)$ i.e. $(\mp 1, \pm 2)$.

100. Let $AD$ be the internal bisector of $\angle BAC$ then $\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$

Thus, $D \equiv \left(\frac{bx_2+cx_3}{b+c}, \frac{by_2+cy_3}{b+c}\right)$

Let the equation of the line $AD$ be $lx+my+n=0$, then we observe that $A$ and $D$ lie on this line. Therefore

$lx_1+my_1+n=0$ and $l\left(\frac{bx_2+cx_3}{b+c}\right)+\frac{m(by_2+cy_3)}{b+c})+n=0$

Eliminating $l, m, n$ gives us

$$\begin{vmatrix} x & y & c_1 \\ x_1 & y_1 & 1 \\ bx_2+cx_3 & by_2+cy_3 & b+c \end{vmatrix} = 0 \Rightarrow b\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + c\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

101. The required points are $\left(1 \pm 6\cos 60^{\circ}, 1 \pm 6\sin 60^{\circ}\right)$ i.e. $\left(4, 1+3\sqrt{3}\right)$ and $\left(-2, 1-3\sqrt{3}\right)$.

102. The equation of the line passing through $(-1,3)$ and slope 1 is given by $y-3=x+1 \Rightarrow x-y+4=0$

Putting $x=y-4$ in the given equation $2y-8+y=3 \Rightarrow y=\frac{11}{3}$ and $x=-\frac{1}{3}$

Distance between $(-1,3)$ $\left(-\frac{1}{3}, \frac{11}{3}\right)$ is $\sqrt{\left(-\frac{1}{3}+1\right)^2+\left(\frac{11}{3}-3\right)^2}=\frac{2\sqrt{2}}{3}$.

103. Let the line through $P(x_1, y_1)$ inclined at angle $\theta$ with the $x$-axis have slope $\tan(\theta)$. Its equation is $y-y_1=\tan\theta(x-x_1)$.

Rewriting, $\tan\theta x - y + (y_1 - x_1\tan\theta) = 0$.

The given line is $ax+by+c=0$.

If $Q$ is the intersection point, the distance $PQ$ measured along the direction making angle $\theta$ with the $x$-axis is $PQ = \left|\frac{ax_1+by_1+c}{a\cos\theta+b\sin\theta}\right|$.

104. Give that the line makes an angle of $30^{\circ}$ with positive direction of $x$-axis and rotated $15^{\circ}$ in anticlockwise direction so the line will now make $45^{\circ}$ with the positive direction of $x$-axis.

Thus, slope of the line is $\tan 45^{\circ}=1$. Also, the line passes through $(2,0)$ so the equation of line is

$y-0=1.(x-2) \Rightarrow x-y-2=0$.

105. Given the line $2x-y=5$. Substitute $y=x$ into the equation: $2x-x=5 \Rightarrow x=5$.

So the point of rotation is $(5,5)$. Slope $m=2$.

After rotation by $45^{\circ}$, the new slope is: $m'=\tan(\arctan(2)+45^{\circ})$.

Using the identity: $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$, we get:

$m'=\frac{2+1}{1-2*1}=\frac{3}{-1}=-3$.

Using point-slope form: $y-5=-3(x-5) \Rightarrow y=-3x+20$.

106. The given line is $x+2y=4$. The line is translated by 3 units in the direction of increasing $x$. So replace $x$ with $x-3$:

$(x-3)+2y=4 \Rightarrow x+2y=7$.

Now the shifted line cuts the $x$-axis at $(y=0)$: $x=7$. So the pivot point is $(7,0)$.

From $x+2y=7$: $y=\left(-\frac{1}{2}\right)x+\frac{7}{2}$, so slope $m=-\frac{1}{2}$.

Angle of inclination $\theta$ satisfies $\tan(\theta)=-\frac{1}{2}$.

After clockwise rotation by $30^{\circ}$, new angle is $\theta-30^{\circ}$.

New slope: $m'=\tan(\theta-30^{\circ})$

$\tan(\theta-30^{\circ})=\frac{\tan(\theta)-\tan(30^{\circ})}{1+\tan(\theta)\tan(30^{\circ})}$

$\Rightarrow m'=\frac{\left(-\frac{1}{2}\right)-\left(\frac{1}{\sqrt{3}}\right)}{1+\left(-\frac{1}{2}\right)\left(\frac{1}{\sqrt{3}}\right)} \Rightarrow -\frac{1}{2}-\frac{1}{\sqrt{3}}=-\frac{\sqrt{3}+2}{2\sqrt{3}}$

$\Rightarrow 1-\frac{1}{2\sqrt{3}}=\frac{2\sqrt{3}-1}{2\sqrt{3}}$. So, $m'=-\frac{\sqrt{3}+2}{2\sqrt{3}-1}$.

Using point-slope form with point $(7,0)$: $y-0=m'(x-7)$

$\Rightarrow y=-\frac{\sqrt{3}+2}{2\sqrt{3}-1}(x-7)$

107. Let the regular hexagon be $ABCDEF$ with side length $a$, and $A$ as origin $(0,0)$. Given $AB$ lies along the $x$-axis and $AE$ along the $y$-axis.

Since $AB=a$ along $x$-axis: $B=(a,0)$

Since $AE=a$ along $y$-axis: $E=(0,a)$

In a regular hexagon, each interior angle is $120^{\circ}$, so directions change by $60^{\circ}$.

Direction $BC$ makes $60^{\circ}$ with $AB$: $C=B+(a\cos 60^{\circ}, a\sin 60^{\circ})=\left(a+\frac{a}{2}, a\frac{\sqrt{3}}{2}\right)=\left(3\frac{a}{2}, \sqrt{3}\frac{a}{2}\right)$

$D=C+(a\cos 120^{\circ}, a\sin 120^{\circ})=\left(3\frac{a}{2}-\frac{a}{2}, \sqrt{3}\frac{a}{2}+\sqrt{3}\frac{a}{2}\right)=\left(a, \sqrt{3}a\right)$

$F=A+(a\cos(-60^{\circ}), a\sin(-60^{\circ}))=\left(\frac{a}{2}, -\sqrt{3}\frac{a}{2}\right)$

Equation of $AC$: $A(0,0)$ and $C\left(3\frac{a}{2}, \sqrt{3}\frac{a}{2}\right)$, $m=\frac{\sqrt{3}\frac{a}{2}}{3\frac{a}{2}}=\frac{\sqrt{3}}{3} \Rightarrow AC: y=\left(\frac{\sqrt{3}}{3}\right)x$

For $AF$: $A(0,0)$ and $F\left(\frac{a}{2}, -\sqrt{3}\frac{a}{2}\right)$, $m=\frac{-\sqrt{3}\frac{a}{2}}{\frac{a}{2}}=-\sqrt{3}$

Equation of $AF: y=-\sqrt{3}x$

For $BE$: $B(a,0)$ and $E(0,a)$, $m=\frac{a-0}{0-a}=-1$

$y-0=-1(x-a) \Rightarrow y=-x+a$

108. Let the place be the origin $O(0,0)$. The road is at a perpendicular distance $5\sqrt{2}$ from $O$, and the shortest distance is in the $N-E$ direction, i.e., along a line making $45^{\circ}$ with the axes.

So the normal to the road has slope 1, hence the road has slope $-1$.

Thus, equation of the road is of form: $y=-x+c$

Distance from origin to this line: $|c\frac{|}{\sqrt{1^2+(-1)^2}}=|c\frac{|}{\sqrt{2}}$

Given distance is $5\sqrt{2}$: $|c\frac{1}{\sqrt{2}} = 5\sqrt{2} \Rightarrow |c| = 10$

Since direction is $N - E$, take positive value: $\Rightarrow c = 10$

So road equation: $y = -x + 10$

(i) Check point $(6, 4)$: $4 = -6 + 10 \Rightarrow 4 = 4$. So, village lies on the road.

(ii) Check point $(4, 3)$: $3 = -4 + 10 \Rightarrow 3 = 6$ So, village does not lie on the road.

109. Given line: $x - y + 1 = 0 \Rightarrow y = x + 1$ so slope $m = 1$

Point of rotation (on $y$-axis): $x = 0 \Rightarrow y = 1$ so $A(0, 1)$

Angle of inclination: $\tan\theta = 1 \Rightarrow \theta = 45^{\circ}$

After clockwise rotation by $75^{\circ}$: new angle $= 45^{\circ} - 75^{\circ} = -30^{\circ}$

New slope: $m' = \tan(-30^{\circ}) = -\frac{1}{\sqrt{3}}$

Equation using point-slope form at $A(0, 1)$: $y - 1 = \left(-\frac{1}{\sqrt{3}}\right)(x - 0) \Rightarrow y = 1 - \frac{x}{\sqrt{3}}$.

110. The diagram is smae as problem 77. $OC$ is 2 units therefore $OB = 2$ units. From the diagram we see that extended $BE$ makes an angle of $45^{\circ}$ with $x$-axis.

Slope: $m = \tan 45^{\circ} = 1$. Equation is intercept form: $y = mx + c \Rightarrow y = x + 2$.

$CD$ will have same slope but passes through $(0, 2)$. Equation in slope-point form: $y - 0 = 1.(x - 2) \Rightarrow x - y = 2$.

111. The midpoint is $\left(\frac{3+1}{2}, \frac{-1+1}{2}\right) = (2, 0)$

Slope: $m = \frac{1-(-1)}{1-3} = \frac{2}{-2} = -1$

A line perpendicular to this will have slope equal to the negative reciprocal of $-1$, which is 1.

Let the required point be $(x, y)$. Since it lies on the perpendicular line passing through $(2, 0)$, its equation is:

$y - 0 = 1(x - 2) \Rightarrow y = x - 2$

Also, the distance from $(2, 0)$ to $(x, y)$ is 2, so: $\mathtt{sqrt}\{(x - 2)^2 + (y - 0)^2\} = 2$

$\Rightarrow (x - 2)^2 + y^2 = 4$. Substitute $y = x - 2$:

$(x - 2)^2 + (x - 2)^2 = 4 \Rightarrow 2(x - 2)^2 = 4 \Rightarrow (x - 2)^2 = 2$

$x - 2 = \pm\sqrt{2} \Rightarrow x = 2 \pm \sqrt{2}$

Then $y = x - 2$ gives: $y = \pm\sqrt{2}$

Since the shift is in the sense of increasing $y$, we take the positive value:

$x = 2 + \sqrt{2}, \quad y = \sqrt{2}$

112. The given line is $2x = y$, i.e., $y = 2x$.

So, the slope of the line is 2. A direction along this line can be taken as $(1, 2)$.

Now, its length is $\sqrt{1^2 + 2^2} = \sqrt{5}$

So, the unit direction along the line is $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

Since the translation is in the first quadrant, both coordinates increase.

Add this to the point $(1, 1)$:

New point $= \left(1 + \frac{1}{\sqrt{5}}, 1 + \frac{2}{\sqrt{5}}\right)$

113. We are given $A(2, -1)$ and the line $x - y = 3$.

Let the required point be $A'(x, y)$. Since the translation is parallel to the line, the slope of the line joining $A$ and $A'$ must be equal to the slope of $x - y = 3$.

Rewrite the line: $y = x - 3$, so slope $= 1$.

Hence, $\frac{y-(-1)}{x-2} = 1 \Rightarrow y + 1 = x - 2 \Rightarrow y = x - 3$.

Now use the distance condition: Distance between $A(2, -1)$ and $A'(x, y)$ is 4.

So, $(x - 2)^2 + (y + 1)^2 = 16$.

Substitute $y = x - 3$: $(x - 2)^2 + ((x - 3) + 1)^2 = 16 \Rightarrow (x - 2)^2 + (x - 2)^2 = 16 \Rightarrow 2(x - 2)^2 = 16 \Rightarrow x - 2 = +-\sqrt{8} = +-2 * \sqrt{2}$.

So, $x = 2 + -2 * \sqrt{2}$. Then, $y = x - 3 = -1 + -2 * \sqrt{2}$.

Thus the two possible points are: $\left(2 + 2 * \sqrt{2}, -1 + 2 * \sqrt{2}\right)$ and $\left(2 - 2 * \sqrt{2}, -1 - 2 * \sqrt{2}\right)$.

114. Both particles start from $A(2, -1)$. First particle moves along the line $x + y = 1$. Rewrite: $y = 1 - x$, so slope $= -1$.

Let its new position be $(x_1, y_1)$. Since it moves towards increasing $y$, we take direction where $y$ increases.

Using slope condition: $\frac{y_1-(-1)}{x_1-2} = -1 \Rightarrow y_1 + 1 = -(x_1 - 2) \Rightarrow y_1 = -x_1 + 1$.

Distance moved is 2, so $(x_1 - 2)^2 + (y_1 + 1)^2 = 4$.

Substitute $y_1 = -x_1 + 1$: $(x_1 - 2)^2 + (-x_1 + 1 + 1)^2 = 4 \Rightarrow (x_1 - 2)^2 + (-x_1 + 2)^2 = 4 \Rightarrow (x_1 - 2)^2 + (x_1 - 2)^2 = 4 \Rightarrow 2(x_1 - 2)^2 = 4 \Rightarrow (x_1 - 2)^2 = 2 \Rightarrow x_1 - 2 = +-\sqrt{2}$.

So, $x_1 = 2 + -\sqrt{2}$. Then $y_1 = -x_1 + 1 = -1 - +\sqrt{2}$.

Since $y$ must increase from $-1$, we take $y_1 = -1 + \sqrt{2}$, hence $x_1 = 2 - \sqrt{2}$.

So first particle's position: $\left(2 - \sqrt{2}, -1 + \sqrt{2}\right)$.

Second particle moves along $x - 2y = 4$. Rewrite: $y = \frac{x-4}{2}$, so slope $= \frac{1}{2}$.

Let position be $(x_2, y_2)$.

Slope: $\frac{y_2-(-1)}{x_2-2} = \frac{1}{2} \Rightarrow y_2 + 1 = \frac{x_2-2}{2} \Rightarrow y_2 = \frac{x_2}{2} - 2$.

Distance moved is 5, so $(x_2 - 2)^2 + (y_2 + 1)^2 = 25$.

Substitute $y_2 = \frac{x_2}{2} - 2$: $(x_2 - 2)^2 + (\frac{x_2}{2} - 2 + 1)^2 = 25 \Rightarrow (x_2 - 2)^2 + (\frac{x_2}{2} - 1)^2 = 25$.

$(x_2 - 2)^2 = x_2^2 - 4x_2 + 4$ $(\frac{x_2}{2} - 1)^2 = \frac{x_2^2}{4} - x_2 + 1$.

So, $x_2^2 - 4x_2 + 4 + \frac{x_2^2}{4} - x_2 + 1 = 25 \Rightarrow (\frac{5}{4})x_2^2 - 5x_2 + 5 = 25 \Rightarrow (\frac{5}{4})x_2^2 - 5x_2 - 20 = 0$ multiply by $4 : 5x_2^2 - 20x_2 - 80 = 0 \Rightarrow x_2^2 - 4x_2 - 16 = 0$.

$\Rightarrow x_2 = \frac{4+-\sqrt{16+64}}{2} = \frac{4+-\sqrt{80}}{2} = 2 + -2 * \sqrt{5}$. Then $y_2 = \frac{x_2}{2} - 2 = -1 + -\sqrt{5}$.

Since $y$ increases from $-1$, take $y_2 = -1 + \sqrt{5}$, so $x_2 = 2 + 2 * \sqrt{5}$.

Thus second particle's position: $\left(2 + 2 * \sqrt{5}, -1 + \sqrt{5}\right)$.

Distance between the two new positions:
$\left[\left(2 + 2 * \sqrt{5}\right) - \left(2 - \sqrt{2}\right)\right]^2 + \left[\left(-1 + \sqrt{5}\right) - \left(-1 + \sqrt{2}\right)\right]^2 = \sqrt{29 + 2 * \sqrt{10}}$.

115. We are given fixed point $A(4, -1)$ and the other end $B(1, 2)$. Let the new position of $B$ after stretching be $B'(x, y)$.

Since the string remains straight, points $A$, $B$, and $B'$ are collinear.
Slope of $AB$: $\frac{2-(-1)}{1-4} = \frac{3}{-3} = -1$.

So equation of line through $A$: $\frac{y-(-1)}{x-4} = -1 \Rightarrow y + 1 = -(x - 4) \Rightarrow y = -x + 3$.

Length of $AB$: $AB^2 = (1 - 4)^2 + (2 + 1)^2 = (-3)^2 + 3^2 = 9 + 9 = 18$.

Since the string is stretched to triple its length, $AB' = 3 * \sqrt{18}$.

Thus, $(x - 4)^2 + (y + 1)^2 = \left(3 * \sqrt{18}\right)^2 = 9 * 18 = 162$.

Substitute $y = -x + 3$: $(x - 4)^2 + (-x + 3 + 1)^2 = 162 \Rightarrow (x - 4)^2 = 81 \Rightarrow x - 4 = \pm 9$.

So, $x = 13$ or $x = -5$. Then $y = -x + 3$: If $x = 13$, $y = -10$. If $x = -5$, $y = 8$.

Now, since the string is stretched beyond $B$, the point $B'$ lies in the same direction from $A$ as $B$.

From $A(4, -1)$ to $B(1, 2)$, $x$ decreases and $y$ increases, so we choose $x = -5$, $y = 8$.

116. On x-axis, $y = 0 \Rightarrow x = 2$. So $A = (2, 0)$. Given $B = (4, 2)$.

Slope of $AB$ is $m = \frac{2-0}{4-2} = 1$. So line $AB$ is $y = x - 2$

The line is rotated anticlockwise by 45° about $A$.

Angle between original line and new line is 45°. So the new angle will be $90^\circ$ i.e. line is parallel to $y$ and passes through $A$ so new line is parallel to $y$-axis i.e. $x = 2$.

After rotation, $B'$ lies on $x = 2$.
Distance $AB = \sqrt{(4 - 2)^2 + (2 - 0)^2} = \sqrt{8} = 2 * \sqrt{2}$.

So: $(x - 2)^2 + (y - 0)^2 = 8$. Since $x = 2$: $y^2 = 8 \Rightarrow y = \pm 2 * \sqrt{2}$

Since rotation is anticlockwise from slope 1, the point moves upward from $A$. So $y > 0$:

$\therefore y = 2 * \sqrt{2}. \Rightarrow B' = \left(2, 2 * \sqrt{2}\right)$

117. Put $x = -1$ in floor equation: $-1 + 2y = 3 \Rightarrow y = 2$. So impact point is $P(-1, 2)$.

   The floor is a straight line, so we use the property:

   Angle of incidence = angle of reflection.

   Incoming path is vertical, so it makes an angle of 90Â° with the x-axis.

   Now find slope of floor: $x + 2y = 3 \Rightarrow y = \frac{3-x}{2}$. So slope of floor is $-\frac{1}{2}$.

   A line perpendicular to floor has slope 2.

   Since incidence is vertical, we consider how a vertical direction reflects across a line of slope $-\frac{1}{2}$.

   The reflected direction must satisfy symmetry about the floor, so we construct it geometrically using slope relation:

   If one direction is vertical, the reflected direction must make equal angle with the floor on the other side. This gives the new slope:

   $m = \frac{3}{4}$. So rebound path passes through $P(-1, 2)$ with slope $\frac{3}{4}$.

   Equation of rebound path: $y - 2 = \left(\frac{3}{4}\right)(x + 1)$

   Height fallen = $2 - 1 = 1$

   Rebound height = $\frac{2}{3}$ So maximum $y$ after rebound: $y = 2 + \frac{2}{3} = \frac{8}{3}$

   Substitute into line: $\frac{8}{3} - 2 = \left(\frac{3}{4}\right)(x + 1) \Rightarrow \frac{2}{3} = \left(\frac{3}{4}\right)(x + 1) \Rightarrow x + 1 = \frac{8}{9} \Rightarrow x = -\frac{1}{9}$

   Since motion is constrained by slanted floor $x + 2y = 3$, the actual highest point must also satisfy proportional displacement along the reflected line segment above the floor.

   Scaling the displacement from $P(-1, 2)$ in ratio consistent with the $2 : 3$ rebound rule along the oblique direction gives:

   $x = -\frac{13}{15}, \quad y = \frac{19}{15}.$

118. Line parallel to $3x - 4y + 1 = 0$ through $A(4, -1)$ is: $3x - 4y + c = 0$

   Substitute $A(4, -1)$: $12 + 4 + c = 0 \Rightarrow c = -16$

   So line is $3x - 4y - 16 = 0$.

   Let a point on it be $(x, y)$ and distance from $A(4, -1)$ be 5: $(x - 4)^2 + (y + 1)^2 = 25$

   From line: $x = \frac{4y+16}{3}$. Substitute: $\left(\frac{4y+16}{3} - 4\right)^2 + (y + 1)^2 = 25$

   $\left(\frac{4y+16-12}{3}\right)^2 + (y + 1)^2 = 25$. So $y = 2$ or $y = -4$

   For $y = 2$: $x = \frac{8+16}{3} = 8$

For $y = -4$: $x = \frac{-16+16}{3} = 0$

119. We measure distance from $P(3,5)$ to the line $2x + 3y = 14$ along a direction parallel to $x - 2y = 1$.

So we move from $(3,5)$ along a line parallel to $x - 2y = 1$ until we meet $2x + 3y = 14$.

A line parallel to $x - 2y = 1$ has form: $x - 2y = k$

Through $P(3,5)$: $3 - 2 * 5 = k \Rightarrow k = -7$

So required line through $P$ is: $x - 2y = -7$

Now find intersection with $2x + 3y = 14$.

From $x - 2y = -7$: $x = 2y - 7$

Substitute: $2(2y - 7) + 3y = 14 \Rightarrow y = 4$

Then: $x = 2 * 4 - 7 = 1$

So intersection point is $Q(1,4)$.

Now distance $PQ = \sqrt{(1-3)^2 + (4-5)^2} = \sqrt{5}$.

120. We measure the distance from $P(2,5)$ to the line $3x + y + 4 = 0$ along a direction parallel to $3x - 4y + 8 = 0$.

A line parallel to $3x - 4y + 8 = 0$ is: $3x - 4y = k$

Through $P(2,5)$: $3(2) - 4(5) = k \Rightarrow 6 - 20 = -14$. So $k = -14$.

Hence required line through $P$ is: $3x - 4y = -14$

Now find its intersection with $3x + y + 4 = 0$.

From $3x - 4y = -14$: $3x = 4y - 14 \Rightarrow x = \frac{4y-14}{3}$

Substitute into $3x + y + 4 = 0$: $3\left(\frac{4y-14}{3}\right) + y + 4 = 0$

$(4y - 14) + y + 4 = 0 \Rightarrow 5y - 10 = 0 \Rightarrow y = 2$

Then: $x = \frac{8-14}{3} = -\frac{6}{3} = -2$. So intersection point is $Q(-2,2)$.

Now distance $PQ = \sqrt{(-2-2)^2 + (2-5)^2} = 5$.

121. Let $A(1,3)$ and $C(5,1)$ be opposite vertices of a rectangle.

Midpoint of diagonal: $M = \left(\frac{1+5}{2}, \frac{3+1}{2}\right) = (3,2)$. So centre is $(3,2)$.

Let other vertices be $B$ and $D$ on line $y = 2x + c$.

Since diagonals bisect each other, $B$ and $D$ are symmetric about $(3,2)$.

So if $B(x,y)$ is on the line, then $D(6-x, 4-y)$ is also on it.

For $B$: $y = 2x + c$. For $D$: $4 - y = 2(6 - x) + c$

Substitute $y$: $4 - (2x + c) = 12 - 2x + c \Rightarrow c = -4$

So line is $y = 2x - 4$. Let $x = 2$, then $y = 0$ so $B(2,0)$.

The other two vertices must lie on this line $y = 2x - 4$ and must be symmetric about the midpoint $M(3, 2)$.

So if we pick any point $B(x, y)$ on the line, its opposite vertex is automatically fixed by midpoint symmetry: $D = (6 - x, 4 - y)$

Now both $B$ and $D$ will always satisfy the line equation, so we are free to choose any value of $x$ that makes calculations simple.

We chose $x = 2$ because it avoids fractions and gives: $y = 2 * 2 - 4 = 0$

So $B = (2, 0)$ is an easy clean point on the line, and then: $D = (6 - 2, 4 - 0) = (4, 4)$.

Thus, $c = -4$.

122. Let the line through $(x', y')$ make an angle $\alpha$ with the x-axis. Its parametric form is:

$x = x' + r\cos\alpha$, $y = y' + r\sin\alpha$, where $r$ is the distance measured along the line.

Substitute into $Ax + By + C = 0$:

$A(x' + r\cos\alpha) + B(y' + r\sin\alpha) + C = 0, Ax' + By' + C + r(A\cos\alpha + B\sin\alpha) = 0$

Solve for $r$: $r(A\cos\alpha + B\sin\alpha) = -(Ax' + By' + C)$

$r = -\frac{Ax'+By'+C}{A\cos\alpha+B\sin\alpha}$

Since length is absolute value of displacement: Length = $|r|$

Length = $\left|\frac{Ax'+By'+C}{A\cos\alpha+B\sin\alpha}\right|$.

123. Let the required line through $P(1, 2)$ be $y - 2 = m(x - 1)$

Parametric form: $x = 1 + t, \quad y = 2 + mt$. So at P, $t = 0$.

Intersection with $x + y - 5 = 0$: $(1 + t) + (2 + mt) - 5 = 0$

$(m + 1)t - 2 = 0 \Rightarrow t_A = \frac{2}{m+1}$

Intersection with $2x - y = 7$: $2(1 + t) - (2 + mt) = 7 \Rightarrow t_B = \frac{7}{2-m}$

Since A and B lie on the same side of P: $t_A, t_B > 0 \Rightarrow -1 < m < 2$

Scale factor along the line is $\sqrt{1 + m^2}$

$PA = \sqrt{1 + m^2}, t_A, \quad PB = \sqrt{1 + m^2}, t_B$

Harmonic mean condition: $= \frac{2}{\frac{1}{PA}+\frac{1}{PB}} = 10$

$= \frac{2}{\frac{1}{st_A}+\frac{1}{st_B}} = 10$, where $s = \sqrt{1 + m^2}$

$= 2\frac{s}{\frac{1}{t_A}+\frac{1}{t_B}} = 10$

$\frac{1}{t_A} = \frac{m+1}{2}, \quad \frac{1}{t_B} = \frac{2-m}{7}$

Sum: $\frac{1}{t_A} + \frac{1}{t_B} = \frac{5m+11}{14}$

$\Rightarrow 2\frac{s}{\frac{5m+11}{14}} = 10$, where $s = \sqrt{1+m^2}$

$14\frac{s}{5m+11} = 5$

Substitute $s = \sqrt{1+m^2}$:

$14\sqrt{1+m^2} = 25m + 55 \Rightarrow 196(1+m^2) = (25m+55)^2$

$196 + 196m^2 = 625m^2 + 2750m + 3025 \Rightarrow 429m^2 + 2750m + 2829 = 0$

Solve: $m = \frac{-2750 \pm \sqrt{2707936}}{858}$

Required line: $y - 2 = m(x-1)$.

124. Any point on the line is $(-2 + r\cos\theta, -3 + r\sin\theta)$. Let $B = (-2 + r_1\cos\theta, -3 + r_1\sin\theta)$ and $C = (-2 + r_2\cos\theta, -3 + r_2\sin\theta)$.

Then $AB = |r_1|$ and $AC = |r_2|$

$\therefore -2 + r_1\cos\theta + 3(-3 + r\sin\theta) = 9 \Rightarrow r_1 = (20)(\cos\theta + 3\sin\theta)$

and $r_2 = \frac{4}{\cos\theta + \sin\theta}$

Given that $AB.AC = 20 \Rightarrow r_1 * r_2 = \pm 20 \Rightarrow \cos^2\theta + 3\sin^2\theta + 4\sin\theta\cos\theta = 4$

$\Rightarrow \tan\theta = 3, 1$. So theh possible lines are $x - y = 1, 3x - y + 3 = 0$

125. Let the line through $P(3,4)$ making angle $\theta$ with the positive x-axis be

$(x,y) = (3,4) + r(\cos\theta, \sin\theta)$

Substitute into $y^2 = 4x$: $(4 + r\sin\theta)^2 = 4(3 + r\cos\theta)$

$\Rightarrow 16 + 8r\sin\theta + r^2\sin^2\theta = 12 + 4r\cos\theta \Rightarrow r^2\sin^2\theta + 8r\sin\theta - 4r\cos\theta + 4 = 0$

$\Rightarrow r^2\sin^2\theta + 4r(2\sin\theta - \cos\theta) + 4 = 0$.

126. The midpoint of $BC$ is $M = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$.

So, the equation of the median from $A$ is the line passing through $A$ and $M$:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \frac{x_2+x_3}{2} & \frac{y_2+y_3}{2} & 1 \end{vmatrix} = 0$$

Using linearity of determinants:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

127. Slope of $x - 2y + 3 = 0$ is $m_1 = -\frac{1}{-2} = \frac{1}{2}$ and slope of $3x + y - 1 = 0$ is $-\frac{3}{1} = -3$.

If $\theta$ is the acute angle between them then $\tan\theta = \frac{|(m_1 - m_2)|}{1 + m_1 m - 2} = 7$.

So the acute angle between them is $\tan^{-1} 7$ and obtuse angle is $\pi - \tan^{-1} 7$.

128. Slope of $x+y=3$ is $m_1=-\frac{1}{1}=-1$. Slope of the line passing through $(1,1)$ and $(-3,4)$ is $m_2=\frac{4-1}{-3-1}=-\frac{3}{4}$.

If $\theta$ is the acute angle between them then $\tan\theta=\frac{|(m_1-m_2)|}{1+m_1m-2}=\frac{1}{7}$.

So the acute angle between them is $\tan^{-1}\frac{1}{7}$ and obtuse angle is $\pi-\tan^{-1}\frac{1}{7}$.

129. Slope of $2x+3y+4+k(6x-y+12)=0$ is $m_1=-\frac{2+6k}{3-k}=\frac{2+6k}{k-3}$ and slope of the line $7x+5y-4=0$ is $m_2=-\frac{7}{5}$.

Since the lines are perpendicular so $m_1m_2=-1\Rightarrow k=-\frac{29}{37}$.

130. Let $A(x_1,y_1),B(x_2,y_2)$ and $C(x_3,y_3)$ be the three vertices of a $ABC$. Let $P$ and $Q$ represent the mid-points of the sides $AB$ and $AC$ respectively.

$P\equiv\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ and $Q\equiv(\frac{x_1+x_3}{2},\frac{y_1+y_3}{2}$

Slope of line $PQ$: $m_1=\frac{\frac{y_1+y_3}{2}-\frac{y_1+y_2}{2}}{\frac{x_1+x_3}{2}-\frac{x_1+x_2}{2}}=\frac{y_3-y_2}{x_3-x_2}$

Slope of line $BC$: $m_2=\frac{y_3-y_2}{x_3-x_2}$).

131. Slope of $AB:m_1=\frac{0-2}{2-0}=-1$. Slope of $CD,m_2=\frac{y-7}{x}$.

Since $AB\parallel CD$, therefore, $m_1=m_2\Rightarrow y=7-x$.

Since the trapezium is isosceles therefore $AD=BC\Rightarrow(x-2)^2+y^2=5\Rightarrow x=7,2\Rightarrow y=0,5$.

132. Slope of the given lines are $m_1=-\frac{a+b}{a-b},m_2=-\frac{a-b}{a+b}$ and $m_3=-1$ respectively.

If angle between first and third line is $\alpha$ then $\tan\alpha=\frac{b}{a}$.

If angle between second and third line is $\beta$ then $\tan\beta=\frac{b}{a}$.

Since both the angles are same we have an isosceles triangle. Let $\theta$ be the vertical angle.

Then $\theta+\alpha+\beta=180^\circ\Rightarrow\theta=180^\circ-2\alpha\Rightarrow\frac{\theta}{2}=90^\circ-\alpha$

$\tan\frac{\theta}{2}=2\cot\alpha\Rightarrow\theta=2\tan^{-1}\frac{a}{b}$.

133. Slope of $x=a$ is $\tan 90^\circ$ and slope of the line is $by+c=0$ is $\tan 0^\circ$. Thus, angle between the two lines is $90^\circ$.

134. Equation of the line whose intercepts are $3,4$ is given by $\frac{x}{3}+\frac{y}{4}=1$ and has a slope of $m_1=-\frac{4}{3}$.

Equation of the line whose intercepts are $1,8$ is given by $\frac{x}{1}+\frac{y}{8}=1$ and has a slope of $m_2=-\frac{8}{1}=-8$

The angle is given by $\theta$ then $\tan\theta=\left|\frac{-\frac{4}{3}+8}{1+\frac{32}{3}}\right|=\frac{4}{7}\Rightarrow\theta=\tan^{-1}\frac{4}{7}$.

135. Slope of the line $\frac{x}{a}+\frac{y}{b}=1$ is $m_1=-\frac{b}{a}$ and slop of the line $\frac{x}{b}-\frac{y}{a}=1$ is $m_2=\frac{a}{b}$.

Clearly, $m_1m_2=-1$, and thus, the two lines are perpendicular to each other.

136. Slope of the line joining $(2, -3)$ and $(-1, 2)$ is $m_1 = \frac{2+3}{-1-2} = -\frac{5}{3}$.

Slope of the line joining $(3, 7)$ and $(-2, 4)$ is $m_2 = \frac{4-7}{-2-3} = \frac{2}{5}$.

Clearly, $m_1 m_2 = -1$, and thus, the two lines are perpendicular to each other.

137. Slope of the line joining $(a, 2a)$ and $(-2, 3)$ is $m_1 = \frac{3-2a}{-2-a}$.

Slope of the line $4x + 3y + 5 = 0$ is $m_2 = -\frac{4}{3}$.

Given that lines are perpendicular to each other therefore $m_1 m_2 = -1$

$\Rightarrow (3 - 2a).(-2 - a).(-\frac{4}{3}) = -1 \Rightarrow 12 - 8a = -6 - 3a \Rightarrow a = \frac{15}{8}$.

138. Both the lines have the same slope of 7, thus lines are parallel to each other.

139. Slope of the line $k^2 x + ky + 1 = 0$ is $-\frac{k^2}{k} = -k$.

Slope of the line $x - ky = 1$ is $\frac{1}{k}$. Clearly, product of the slopes is $-1$ i.e. lines are perpendicular to each other.

140. Slope of the line $x - y + 2 + k(2x + 3y) = 0$ is $-\frac{2k+1}{3}k - 1 \Rightarrow \frac{2k+1}{1-3k}$.

Slope of the line $3x + y = 0$ is $-3$.

Because the lines are parallel the slopes will be equal. $\Rightarrow 2k + 1 = 9k - 3 \Rightarrow k = \frac{4}{7}$.

141. First and third lines have same slope, and, second and fourth lines have same slope. Thus, they will form a parallelogram.

142. Slope of the line $x \cos\theta + y \sin\theta = 2$ is $m_1 = -\frac{\cos\theta}{\sin\theta} = -\cot\theta$.

Slope of the line $x - y = 3$ is $m_2 = 1$.

Since the lines are perpendicular $m_1 m_2 = -1 \Rightarrow \cot\theta = 1 \Rightarrow \theta = 45^{\circ}$.

143. Slope of the line $x - 3y + 5 + k(x + y - 3) = 0$ is $m_1 = \frac{k+1}{3} - k$. Slope of the line $x + y = 1$ is $-1$.

Since the lines are perpendicular $\Rightarrow m_1 m_2 = -1 \Rightarrow k + 1 = k + 1 = 3 - k \Rightarrow k = 1$.

Thus, equation of the first line becomes $2x - 2y + 2 = 0 \Rightarrow x - y + 1 = 0$.

144. Let $A = (0, 0), B = (a, 0), C = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$. (You can get these points by rotating the line moving by a distance $a$ along that line)

Midpoint of $AB$: $M = \left(\frac{0+a}{2}, \frac{0+0}{2}\right) = \left(\frac{a}{2}, 0\right)$

Slope of $AB$: $m_{AB} = \frac{0-0}{a-0} = 0$

Since $x_C = x_M = \frac{a}{2}$, the line CM is vertical.

A vertical line is perpendicular to a horizontal line($AB$).

Hence $CM \perp AB$.

145. Place the rhombus with vertices $A = (a, 0)$, $B = (0, b)$, $C = (-a, 0)$, $D = (0, -b)$ for $a, b > 0$.

Each side has length $|AB| = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2+b^2}$, and by symmetry all four sides are equal, confirming $ABCD$ is a rhombus.

The diagonal $AC$ runs from $(a, 0)$ to $(-a, 0)$, so its slope is $m_{AC} = \frac{0-0}{-a-a} = 0$.

The diagonal $BD$ runs from $(0, b)$ to $(0, -b)$, so it is a vertical line with undefined slope, meaning it is parallel to the $y$-axis.

A line with slope 0 is horizontal, and a vertical line is perpendicular to every horizontal line. Therefore $AC \perp BD$.

146. Equation of the line parallel to the given line is $3x - y + k = 0$. Also, given that this passes through $(3, 4)$; putting the point in the equation

$3 * 3 - 4 + k = 0 \Rightarrow k = -5$.

Thus, equation of the required line is $3x - y - 5 = 0$.

147. The line perpendicular to the line $4x - 3y = 10$ is given by $3x + 4y + k = 0$.

Given that it passes through $(2, 3)$, putting this point in the equation

$2 * 3 + 4. * 3 + k = 0 \Rightarrow k = -18$.

Thus, equation of the required line is $3x + 4y - 18 = 0$.

148. Since the intercept is $\frac{4}{3}$ on $y$ axis therefore the line passes through $\left(0, \frac{4}{3}\right)$. Also lines perpendicular to the line $3 - 4y + 11 = 0$ is given by

$4x + 3y + k = 0$. Putting $\left(0, \frac{4}{3}\right)$ gives us $k = -4$.

Thus, required line is $4x + 3y - 4 = 0$.

149. Mid-point of $(1, 1)$ and $(2, 3)$ is $\left(\frac{3}{2}, 2\right)$. Equation of the line passing though these two points is

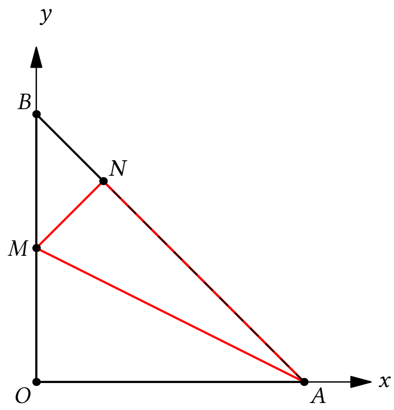
$y - 1 = \frac{3-1}{2-1}(x - 1) \Rightarrow y - 1 = 2x - 2 \Rightarrow 2x - y = 0$.

Line perpendicular to this wll be $x + 2y + k = 0$, which passes through the mid-point. Thus,

$\frac{3}{2} + 4 + k = 0 \Rightarrow k = -\frac{11}{2}$.

Thus, equation of the line is $2x + 4y - 11 = 0$.

150.




Given line is $x+y=a$, which cuts the $x$ and $y$ axes at $A$ and $B$ respectively. $\Rightarrow A \equiv (a,0)$ and $B \equiv (0,a)$.

Let $\frac{AN}{NK} = k$, then $N = \left(\frac{a}{1+k}, \frac{ka}{1+k}\right)$.

Since line $MN$ is perpendicular to $AB$ and it passes through $N$ therefore $x - y - \left(\frac{a}{1+k} - \frac{ka}{1+k}\right) = 0 \Rightarrow x - y = \frac{1-k}{1+k}a$

This line cuts the $y$-axis at $M$. THus, $M \equiv \left(0, \frac{k-1}{k+1}a\right)$

$\Delta AMN = \frac{1}{2}.AN.NM$

$AN^2 = \left(a - \frac{a}{k+1}\right)^2 + \left(0 - \frac{ka}{1+k}\right)^2 \Rightarrow AN = \frac{\sqrt{2}ak}{1+k}$

$NM = \frac{\sqrt{2}a}{1+k}$.

Thus, $\Delta AMN = \frac{a^2k}{(1+k)^2}$ and $\Delta OAB = \frac{1}{2}.a^2$

Thus, $\Delta AMN : \Delta OAB = 3:8 \Rightarrow 3(1+k)^2 = 16k \Rightarrow k = 3, \frac{1}{3}$ but if $k = \frac{1}{3}$ then $M$ lies outside of $OB$. Thus, $k = 3$.

151. Slope of the line $3x - y + 5 = 0$ is 3. Let the slope of the required line is $m$ then $\tan 45^\circ = \left|\frac{m-3}{1+3m}\right| \Rightarrow \frac{m-3}{1+m} = \pm 1 \therefore m = -2, \frac{1}{2}$.

152. Slope of the given line is $\frac{1}{2}$. Let $m$ be the slope of the line passing through $(3,2)$ and making an angle of $45^\circ$ with the line

$\Rightarrow \tan 45^\circ = \frac{|\frac{1}{2} - m|}{1 + \frac{m}{2}} \Rightarrow m = 3, -\frac{1}{2}$.

Thus, equations of required line are $y - 2 = 3(x-3)$ and $y - 2 = -\frac{1}{2}(x-3)$ i.e. $3x - y = 7$ and $x + 3y = 9$.

153. Given line is $x + y - 2 = 0$, its slope, $m_1 = -1$.

Let the slope of the line which makes an angle of $60^\circ$ with this line be $m_1$, then

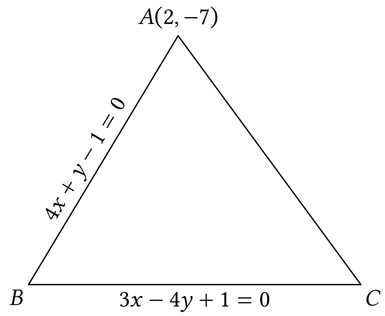
$\tan 60^\circ = \left|\frac{m_1 - m}{1 + m.m_1}\right| \Rightarrow \sqrt{3} = \left|\frac{-1-m}{1-m}\right|$

$m = 2+\sqrt{3}, 2-\sqrt{3}$.

Thus, equation of two other sides of the triangle are

$y-3 = \left(2+\sqrt{3}\right)(x-2)$ and $y-3 = \left(2-\sqrt{3}\right)(x-2)$.

154.




Given that equation of $BC$ is $3x-4y+1=0$ and the equation of $AB$ is $4x+y-1=0$.

Since $AB = AC$, therefore, $\angle ABC = \angle ACB = \alpha$(say)

Slope of the line $BC = \frac{3}{4}$ and slope of $AB = -4$. Let slope of $AC = m$.

Thus, $\frac{-4-\frac{3}{4}}{1-\frac{4.3}{4}} = \frac{\frac{3}{4}-m}{1+\frac{2}{4}m} \Rightarrow m = -\frac{52}{89}$

Thus, equation of $AC$ is $y+7 = -\frac{52}{89}(x-2) \Rightarrow 52x+89y+519 = 0$.

155. Equation of line through $(-2,-7)$ is given by $y+7 = m(x+2) \Rightarrow y = mx + 2m - 7$. This line cuts the given lines at $A$ and $B$ respectively. Solving the equations gives

$A = \left(\frac{33-6m}{4+3m}, \frac{20m-28}{4+3}\right)$ and $B = \left(\frac{24-6m}{4+3m}, \frac{11m-28}{4+3m}\right)$

According to question $AB = 3 \Rightarrow AB^2 = 9 \Rightarrow \frac{81}{(4+3m)^2} + \frac{81m^2}{(4+3m)^2} = 9$

$\Rightarrow 9 + 9m^2 = 16 + 9m^2 + 24m \Rightarrow m = -\frac{7}{24}$

When $m \to \infty$ both sides become tend to 9 i.e. line may be perpendicular to $x$ -axis.

Thus, equations of the required lines are $x = -2$ and $y+7 = -\frac{7}{24}(x+2) \Rightarrow 7x + 24y + 182 = 0$.

156. The line parallel to $x+2y = 3$ is given by $x+2y = k$. Since it passes through $(3,4)$, therefore

$k = 3 + 2*4 = 11$. Hence, equation of the required line becomes $x+2y = 11$.

157. The line parallel to $3x+4y = 12$ is given by $3x+4y = k$. Since it passes through $(4,3)$, therefore

$k = 3*4 + 4*3 = 24$. Hence, equation of the required line becomes $3x+4y = 24$.

158. Equation of the straight line parallel to $3x - 4y + 6 = 0$ is given by $3x - 4y + k = 0$. It passes through the mid-point of the line segment made by $(2, 3)$ and $(4, -1)$ i.e. $(3, 1)$.

Thus, $3 * 3 - 4 * 1 + k = 0 \Rightarrow k = -5$. Hence, the equation of the required line is $3x - 4y - 5 = 0$.

159. Euation of the line joining the points $(2, 3)$ and $(3, -1)$ is given by

$y - 3 = \frac{-1-3}{3-2}(x - 2) \Rightarrow y - 3 = 8 - 4x \Rightarrow 4x + y = 11$

A line parallel to above line wil be $4x + y = k$. Since it passes through $(2, 1)$, therefore,

$4 * 2 + 1 = k \Rightarrow k = 9$. So the required line becomes $4x + y = 9$.

160. Equation of the line parallel to the line $lx + my + n = 0$ is given by $lx + my + k = 0$.

Since it passes through $(\alpha, \beta)$, therefore, $l\alpha + m\beta + k = 0$.

Thus, equation of required line is $lx + my - (l\alpha + m\beta) = 0$.

161. Equation of the line perpendicular to the line $2x + 5y = 31$ is $5x - 2y + k = 0$. Since it passes through $(2, 5)$, therefore,

$5 * 2 - 2 * 5 + k = 0 \Rightarrow k = 0$. So the required line is $5x - 2y = 0$.

162. Any line perpendivular to the given line is $2ay + xy' + k = 0$. Since it passing through $(x', y')$, therefore,

$k = -2ay' - x'y'$. Thus, required line is $2a(y - y') + y'(x - x') = 0$.

163. Slope of the first line is $m_1 = \frac{mn+n^2}{m^2-mn}$, and the slope of the second line is $m_2 = \frac{mn-n^2}{mn+m^2}$.

Let $\theta$ be the angle between these lines then $\tan\theta = \left|\frac{\frac{mn+n^2}{m^2-mn} - \frac{mn-n^2}{mn+m^2}}{1+\frac{mn+n^2}{m^2-mn}\frac{mn-n^2}{mn+m^2}}\right|$

$\Rightarrow \tan\theta = \frac{4m^2n^2}{m^4-n^4} \Rightarrow \theta = \tan^{-1}\frac{4m^2n^2}{m^4-n^4}$.

164. Any line perpendicular to the line $x\sec\theta + y\csc\theta = a$ is given by $x\csc\theta - y\sec\theta = k$, however, the line passes through $\left(a\cos^3\theta, a\sin^3\theta\right)$, therefore,

$a\cos^3\theta\csc\theta - a\sin^3\theta\sec\theta = k = \frac{a\cos^4\theta - a\sin^4\theta}{\sin\theta\cos\theta} = \frac{a\cos 2\theta}{\sin\theta\cos\theta}$

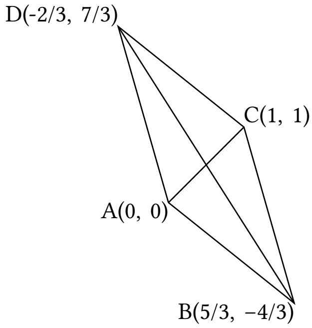
Thus, the given line becomes $x\cos\theta - y\sin\theta = a\cos 2\theta$.

165. Any line perpendicular to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is given by $\frac{x}{b}\sin\theta - \frac{y}{a}\cos\theta = k$. Since this new line passes through $(a\cos\theta, b\sin\theta)$, therefore

$k = \frac{a\cos\theta * \sin\theta}{b} - \frac{b\sin\theta * \cos\theta}{a}$. Thus, equation of the new line becomes

$ax\sec\theta - by\csc\theta = a^2 - b^2$.

166. Let the parallelogram be $ABCD$.




Let the equations of sides $AB$ and $AD$ of the parallelogram be $4x+5y=0$ and $7x+2y=0$. Solving these equations gives $A=(0,0)$.

Equation of one of the diagonals of the parallelogram is $11x+7y=9$, which does not pass through $A$ so it must be the diagonal $BD$.

Solving $AD$ and $BD$ and $AB$ and $BD$ gives us $B=\left(\frac{5}{3},-\frac{4}{3}\right)$ and $D=\left(-\frac{2}{3},\frac{7}{3}\right)$,

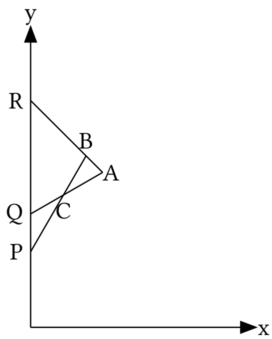
Thus, mid-point of diagonals is $H=\left(\frac{1}{2},\frac{1}{2}\right)$. Thus, equation of the other diagonal which passes through $A$ and $H$ is $x=y$.

167. Solving the three equations pairwise gives us three coordinates $A\equiv=\left(\frac{c-c_1}{m_1-m_2},\frac{m_1c_2-m_2c_1}{m_1-m_2}\right), B\equiv(0,c_1)$ and $C\equiv(0,c_2)$.

Putting these points in the formula for area of triangle $|\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_1)]|$

$=\frac{1}{2}.\frac{c_2-c_1}{m_1-m_2}.(c_1-c_2)=\frac{1}{2}.\frac{(c_2-c_1)^2}{m_1.m_2}.$

168.




Let the three lines be $BC, CA$ and $AB$ whose equations are $y=m_1x+c_1, y=m_2x+c_2$ and $y=m_3x+c_3$.

Let the lines $BC, CA$ and $AB$ meet $y$-axis at $P, Q$ and $R$ respectively. From figure

$\Delta ABC=\Delta AQR-\Delta BPR+\Delta CPQ$

Proceeding like previous problem we have the required result.

169. Let $A(x_1, y_1)$ be the point of intersection of first two equations, $B(x-2, y_2)$ that of second and third, and $C(x_3, y_3)$ that of first and last equations.

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$= \frac{1}{2}\begin{vmatrix} a_1x_1+b_1y_1+c_1 & a_2x_1+b_2y_1+c_2+a_3x_1+b_3y_1+c_3 \\ a_1x_2+b_1y_2+c_1 & a_2x_2+b_2y_2+c_2+a_3x_2+b_3y_2+c_3 \\ a_1x_3+b_1y_3+c_1 & a_2x_3+b_2y_3+c_2+a_3x_3+b_3y_3+c_3 \end{vmatrix} \div \Delta$$

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & a_3x_1+a_3y_1+c_3 \\ a_1x_2+b_1y_2+c_2 & 0 & 0 \\ 0 & a_2x_3+b_2y_3+c_2 & 0 \end{vmatrix} \div \Delta$$

$(x_1, y_1)$ satisdied the above equation and also $a_1x_1 + b_2y_1 + c_1 = 0$ and $a_2x_1 + b_2y_1 + c_2 = 0$

Let $a_3x_1 + b_3y_1 + c_3 = \lambda_1 \Rightarrow a_3x_1 + b_3y_1 + c_3 - \lambda = 0$

Thus, eliminating $(x_1, y_1)$ from these equations gives us

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3-\lambda_1 \end{vmatrix} = 0 \Rightarrow \Delta - \lambda_1\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$\Rightarrow \Delta - \lambda_1 C_3 = 0 \Rightarrow \lambda_1 = \frac{\Delta}{C_3}$, where $C_3$ is cofactor of $c_1$. Similarly, $\lambda_2 = \frac{\Delta}{C_1}$ and $\lambda_3 = \frac{\Delta}{C_2}$

Thus, area of the required triangle is $\left|\frac{1}{2}(\lambda_1\lambda_2\lambda_3) \div \Delta\right| = \frac{\Delta^2}{2|C_1C_2C_3|}$.

170. The equation of any line with gradient 2 is given by $y = 2x + c$. This line will intersect with given lines at $A \equiv \left(\frac{3}{2} - \frac{c}{6}, 3 + \frac{2c}{3}\right), B \equiv \left(1 - \frac{2c}{3}, 2 - \frac{c}{3}\right)$ and $C \equiv \left(2 + \frac{c}{3}, 4 + \frac{5c}{3}\right)$.

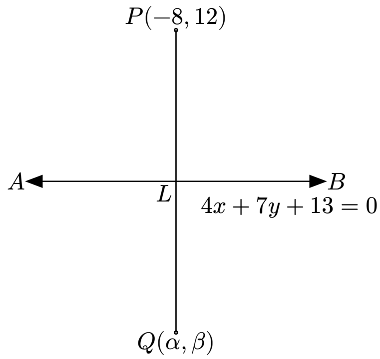
Clearly, $A$ is the middle point of $BC$. Thus, intercepts are equal.

171. Let $(p, q)$ be the foot of the perpendicular. Then $q = 3p + 4$. Also, line perpendicular to it will have the equation $x + 3y = k$.

Since $x + 3y = k$ will pass through $(2, 3)$, therefore, $k = 11$. Also, $(p, q)$ will lie on this line so $p + 3q = 11$.

Solving the two equations gives us $(p, q) = \left(-\frac{1}{10}, \frac{37}{10}\right)$.

172.




Equation of the line mirror $AB$ is $4x + 7y + 13 = 0$. Let $P \equiv (-8, 12)$ and $Q \equiv (\alpha, \beta)$ be the image of $P$ in the line mirror $AB$. Then $PQ \perp AB$ and $PL = LQ$.

Thus equation of $PL$ would be $7x - 4y = k$. Since it passes through $P(-8, 12)$, therefore,

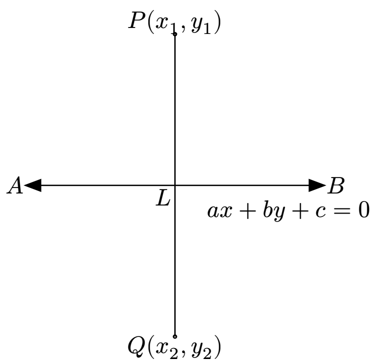
$k = -56 - 48 = -104$. So the equation of $PL$ is $7x - 4y + 104 = 0$. Solving the equations $L \equiv (-12, 5)$.

Since $Q$ is mirror image of $P$ so $L$ will be midpoint of $Q$. Thus, $Q \equiv (-16, -2)$.

173. Refere to Figure 3.15. Slope of $AB = -\frac{a}{b}$ and that of $PQ = \frac{y_2 - y_1}{x_2 - x_1}$.

Since $AB \perp PQ \Rightarrow \left(-\frac{a}{b}\right).\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = -1 \Rightarrow \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = k(\text{say})$

$\Rightarrow x_2 - x_1 = ka, y_2 - y_1 = kb \Rightarrow a(x_2 - x_1) + b(y_2 - y_1) = k(a^2 + b^2)$



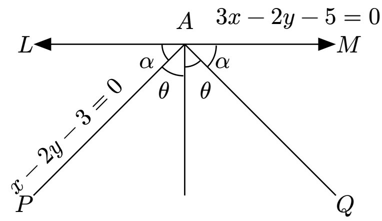

Like previous problem $L$ is the midpoint of $PQ$ i.e. $L \equiv \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ and it also lies on the line $AB$, therefore

$a\left(\frac{x_1 + x_2}{2}\right) + b\left(\frac{y_1 + y_2}{2}\right) + c = 0 \Rightarrow ax_2 + by_2 + c = -(ax_1 + by_1 + c)$

From previously obtained equation $-2(ax_1 + by_1 + c) = k(a^2 + b^2)$

Thus, $\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{2(ax_1 + by_1 + c_1)}{a^2 + b^2}$.

174.




Given equation of horizontal line is $3x-2y-5=0$ and equation of $PA$ is $x-2y-3=0$.

Solving these two equations gives us $A \equiv (1,-1)$.

Let slope of $AQ = m$. Slope of horizontal line is $\frac{3}{2}$ and slope of $AP = \frac{1}{2}$.

Let $\angle LAP = \alpha$ then $\angle QAM = \alpha$.

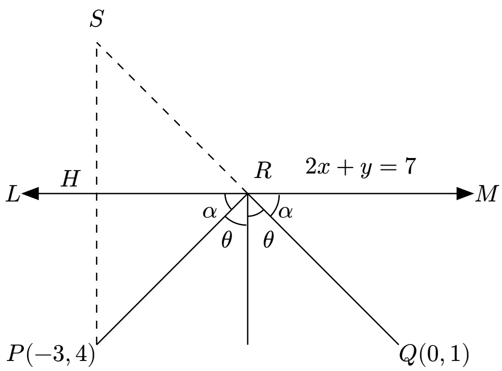
$\therefore \tan\alpha = \frac{4}{7}, \left|\frac{2,m-3}{2+3m}\right|$

Thus, $m = \frac{1}{2}, \frac{29}{2}$. So slope of $AQ = \frac{29}{2}$, and hence, equation of $AQ$ is given by

$y+1 = \frac{29}{2}(x-1) \Rightarrow 29x-2y-31 = 0.$

175. Since the light travels through the shorted path $PR$ must be the incident ray and $RQ$ should be the reflected ray. If $S$ be the image of $P$ w.r.t. line mirror $2x+y=7$, then $PR+RQ = SR+RQ$.

Thus, $PR+RQ$ will be least when $SR+RQ$ will be least i.e. when point $Q, R, S$ are collinear.



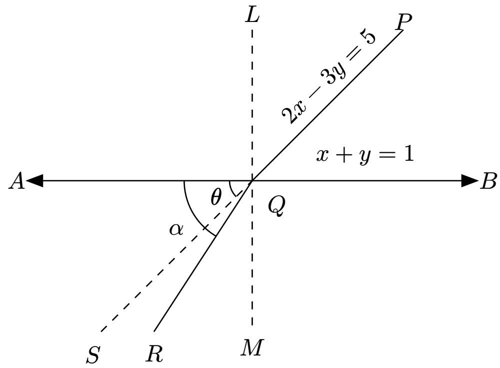

GIven equation of the line $LM$ is $2x+y=7$. Equation of $PH$ would be $x-2y=k$, which passes through $(-3,4)$ so $k=-3-8=-11$

Solving the two equations gives $H \equiv \left(\frac{3}{5}, \frac{29}{5}\right)$. Let $S \equiv (\alpha, \beta)$ then

$\frac{3}{5} = \frac{\alpha-3}{5} \Rightarrow \alpha = \frac{21}{5}$ and $\beta = \frac{38}{5}$

Equation of $SQ$ is $y-1 = \frac{\frac{38}{5}-1}{\frac{21}{5}}x \Rightarrow 11x-17y+7=0.$

176.




$QR$ is the refracted ray. According to question, $\angle SQR = 15^\circ$. From given equations we get $Q \equiv \left(\frac{8}{5}, -\frac{3}{5}\right)$

Slope of $QP = \frac{2}{3}$, so slope of $QS = \frac{2}{3}$. Slope of $AB = -1$

Let $\angle PQB = \theta = \angle AQS$, then $\tan\theta = \left|\frac{\frac{2}{3}+1}{1-\frac{2}{3}}\right| = 5$

Let slope of $QR = m$. $\because \angle SQR = 15^\circ \therefore \tan 15^\circ = \left|\frac{\frac{2}{3}-m}{1+\frac{2}{3}m}\right| = 2-\sqrt{3} \Rightarrow m = \frac{3\sqrt{3}-4}{7-2\sqrt{3}}, \frac{3\sqrt{3}-8}{1-2\sqrt{3}}$.

177. Any line passing through the point of intersection of given lines is given by $5x - y - 9 + k(x + 6y - 8) = 0$.

Given that this line passes through $(2, -2)$, then $2 + 2 - 9 + k(2 - 12 - 8) = 0 \Rightarrow k = \frac{1}{6}$.

So by putting $k$ back in the equaiotn to obtain $x - 2 = 0$.

178. Equation of the line passing through the point of intersection of given lines is $x - y - 1 + k(2x - 3y + 1) = 0$

Slope of this line is $\frac{1+2k}{1+3k}$. Given that this new line is parallel to $3x + 4y - 14 = 0$

Thus, $\frac{1+2k}{1+3k} = -\frac{3}{4} \Rightarrow k = -\frac{7}{17}$. Putting this value of $k$ in the equation above gives us the required equation as $3x + 4y - 24 = 0$.

179. Equation of the line passing through the point of intersection of given lines is $3x - 4y - 7 + k(12x - 5y - 13) = 0$

Slope of this line is $\frac{3+12k}{4+5k}$. Given that this new line is perpendicular to $2x - 3y + 5 = 0$ so the new line's slope must be equal to $\frac{3}{2}$.

Thus, $\frac{3+12k}{4+5k} = \frac{3}{2} \Rightarrow k = -\frac{6}{13}$. Putting this value of $k$ in the equation above gives us the required equation as $33x + 22y + 13 = 0$.

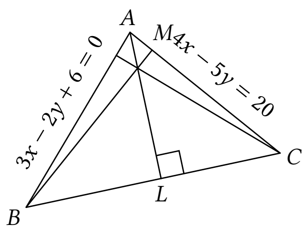
180. Equation of the line passing through the point of intersection of given lines is $x + 3y + 4 + k(3x + y + 4) = 0$.

Slope of this line $-\frac{1+3k}{3+k}$. We know that slope of the lines equally inclined to axes are $\pm 1$.

Equating: $-\frac{1+3k}{3+k} = \pm 1 \Rightarrow k = -1, 1$

Thus, required lines are $x - y = 0$ and $x + y + 2 = 0$.

181. Let the equation of line $AB$ and $AC$ be $3x - 2y + 6 = 0$ and $4x + 5y - 20 = 0$.




Since $BM$ passes through the orthocenter $H(1,1)$ and is perpendicular to $AC$, therefore, equation of $BM$ will be $5x - 4y - (5 * 1 - 4 * 1) = 0 \Rightarrow 5x - 4y - 1 = 0$

Similarly equation of $CN$ will ve $2x + 3y - 5 = 0$

Solving $AB$ and $BM$ gives us $B \equiv \left(-13, -\frac{33}{2}\right)$ and solving $AC$ and $CN$ gives $C \equiv \left(\frac{35}{2}, -10\right)$.

Thus, equation of $BC$ will ve $26x - 122y - 1675 = 0$.

182. Let $ABCD$ be a parallelogram such that $AB$ is $u = p, BC$ is $v = s, CD$ is $u = q$, and $AD$ is $v = r$.

Equation of $AC$, which passes through the point of intersection of lines $u - p = 0$ and $v - r = 0$ is

$u - p + k(v - r) = 0 \Rightarrow ax + by + c - p + k(a'x + b'y + c' - r) = 0$

Let $C \equiv (\alpha, \beta)$, then $C$ lies on $AC$, therefore,

$a\alpha + b\beta + c - p + k(a'\alpha + b'\beta + c' - r) = 0$ but $u = q$ and $v = s$

So the line becomes $q - p + k(s - r) = 0 \therefore k = \frac{p-q}{s-r}$

$\Rightarrow u - p + \frac{p-q}{s-r}(v - r) = 0 \Rightarrow u(r - s) - v(p - q) + ps - qr = 0$

$\Rightarrow \begin{vmatrix} u & v & 1 \\ p & r & 1 \\ q & s & 1 \end{vmatrix} = 0.$

183. We can write the given equation as $a(x + y - 1) + b(2x + 3y - 1) = 0$. Clearly, both $a$ and $b$ cannot be zero at the same time. Let $a \neq 0$, then

$x + y - 1 + \frac{b}{a(2x+3y-1)} = 0$. This line passes through the point of intersection of $x + y - 1 = 0$ and $2x + 3y - 1 = 0$, i.e. $x = 2, y = -1$.

Thus, all given straight lines pass through the fixed point $(2, -1)$.

184. Given that $al + bm + cn = 0 \Rightarrow n = -\frac{al+bm}{c}$. Putting this in the equation of line

$lx + my - \frac{al+bm}{c} = 0 \Rightarrow l(cx - a) + m(cy - b) = 0$

Clearly, both $l$ and $m$ both can be zero because in the equation of straight line coeff. of both $x$ and $y$ cannot be zero.

Thus, the give equation represents straight lines which passes through the intersection of lines $cx - a = 0$ and $cy - b = 0$ i.e. through the point $\left(\frac{a}{c}, \frac{b}{c}\right)$.

185. Let $O$ the point of intersections of the lines be origin then we can represent the equations as $y = m_r x, r = 1, 2, 3, ..., n$.

Let the variable line be $y = mx + c$. Solving gives us $x = \frac{c}{m_r - m}$ and $y = \frac{m_r c}{m_r - m}$

$OA_r^2 = \frac{c^2}{(m_r - m)^2}(1 + m_r^2) \Rightarrow OA_r = \left|\frac{c}{m_r - m}\right|\sqrt{1 = m_r^2} \Rightarrow \frac{1}{OA_r} = \left|\frac{m_r - m}{c}\right|\frac{1}{\sqrt{1+m_r^2}}$

$$\Rightarrow \sum_{i=1}^{n} \frac{1}{OA_i} = \pm\frac{m_r - m}{c}.\frac{1}{\sqrt{1+m_r^2}} = k$$

Thus, $y = mx + c$ passes through the fixed point $\left(\frac{1}{k}\sum \pm \frac{1}{\sqrt{1+m_r^2}}, \frac{1}{k}\sum \pm \frac{m_r}{\sqrt{1+m_r^2}}\right)$.

186. $\Delta = \begin{vmatrix} 4 & 7 & -9 \\ 5 & -8 & 15 \\ 9 & -1 & 6 \end{vmatrix} = 0.$

187. Let $ABC$ be a triangle with vertices $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$.

The median through $A$ will pass through $A(x_1, y_1)$ and $D\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$. The median through $B$ will pass through $B$ and $E\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$. The median through $B$ will pass through $B$ and $F\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Equation of $AD$ is given by $y - y_1 = \frac{\frac{y_2+y_3}{2} - y_1}{\frac{x_2+x_3}{2} - x_1}(x - x_1)$

$\Rightarrow (2y_1 - y_2 - y_3)x - (2x_1 - x_2 - x_3)y - x_1(2y_1 - y_2 - y_3) + y_1(2x_1 - x_2 - x_3) = 0$

Similarly we can find the equation of $BE$ and $CF$.

We find that sum of equations is $0 = 0$. Thus, medians are concurrent.

188. By trial we observe that $(p + q)x + (p + q)y - (p - q) + (p - q)x - (p - q)y - (p + q) + 2[px + qy - p] = 0$

Thus, first three lines are concurrent. We also see that

$(p - q)x - (p - q)y - (p + q) + px + qy - p + qx + py + q = 0.$

Thus, last three lines are concurrent making all four lines concurrent.

189. The lines are concurrent therefore $\begin{vmatrix} p_1 & q_1 & -1 \\ p_2 & q_1 & -1 \\ p_3 & q_3 & -1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} p_1 & q_1 & 1 \\ p_2 & q_1 & 1 \\ p_3 & q_3 & 1 \end{vmatrix} = 0$, and hence, the three points are concurrent.

190. Subtract the first equation from the second equation $(x+2y)-(x-4y)=9-3 \Rightarrow 6y=6 \Rightarrow y=1$

Substitute into $x+2y=9, x+2(1)=9 \Rightarrow x=7$

So the intersection point is $(7,1)$.

Now substituting into the line $mx+2y+5=0, m(7)+2(1)+5=0 \Rightarrow 7m+7=0 \Rightarrow m=-1$.

191. Subtracting yields $y(t_1-t_2)=a(t_1^2-t_2^2) \Rightarrow y=a(t_1+t_2) \Rightarrow x=at_1t_2$.

192. From $x+y=3$, we get $x=3-y$. Substitute into the second equation, $2(3-y)-3y=1 \Rightarrow y=1$

Then $x=3-1=2$. So the point of intersection is $(2,1)$.

Now the line is $\frac{x}{a}+\frac{y}{b}=1$ and it passes through $(2,1)$:

$\Rightarrow \frac{2}{a}+\frac{1}{b}=1$

Rewrite the line: $\frac{x}{a}+\frac{y}{b}=1 \Rightarrow y=b-\left(\frac{b}{a}\right)x$. So slope $=-\frac{b}{a}$.

Given it is parallel to $y=x-6$, whose slope is 1:

$-\frac{b}{a}=1 \Rightarrow b=-a$

Substitute into the earlier equation: $\frac{2}{a}+\frac{1}{-a}=1 \Rightarrow a=1$. Then $b=-1$.

193. Intersection of $x=y$ and $y=2x$: $x=2x \Rightarrow x=0, y=0$

Intersection of $x=y$ and $y=3x+4$: $x=3x+4 \Rightarrow -2x=4 \Rightarrow x=-2, y=-2$

Intersection of $y=2x$ and $y=3x+4$: $2x=3x+4 \Rightarrow -x=4 \Rightarrow x=-4, y=-8$

Vertices: $(0,0),(-2,-2),(-4,-8)$

Area=$\frac{1}{2}*|0(-2+8)+(-2)(-8-0)+(-4)(0+2)|=\frac{1}{2}*|0+16-8|=4$

194. From $3x-4y+4a=0$ and $2x-3y+4a=0$: $3x-4y=-4a, 2x-3y=-4a \Rightarrow 6x-8y=-8a, 6x-9y=-12a$

$\Rightarrow y=4a$. Substituteing, $2x-3(4a)=-4a \Rightarrow 2x-12a=-4a \Rightarrow 2x=8a \Rightarrow x=4a$. Point:$(4a,4a)$.

From $2x-3y+4a=0$ and $5x-y+a=0$: $2x-3y=-4a, 5x-y=-a$

From second: $y=5x+a$. Substituting: $2x-3(5x+a)=-4a \Rightarrow 2x-15x-3a=-4a \Rightarrow -13x=-a \Rightarrow x=\frac{a}{13}$

$y=5\left(\frac{a}{13}\right)+a=\frac{5a}{13}+\frac{13a}{13}=\frac{18a}{13}$. Point:$\left(\frac{a}{13},18\frac{a}{13}\right)$.

From $3x-4y+4a=0$ and $5x-y+a=0$. Point:$(0,a)$

Vertices:$(4a,4a),\left(\frac{a}{13},\frac{18a}{13}\right),(0,a)$

Area=$\frac{1}{2}*\mathsf{abs}\left[4a\left(\frac{18a}{13}-a\right)+\left(\frac{a}{13}\right)(a-4a)+0\left(4a-\frac{18a}{13}\right)\right]=\frac{17a^2}{26}$.

195. Intersection points: $y = m_1x$ and $y = m_2x$ give $(0, 0)$

$y = c$ with $y = m_1x$ gives $\left(\frac{c}{m_1}, c\right)$

$y = c$ with $y = m_2x$ gives $\left(\frac{c}{m_2}, c\right)$

So triangle vertices are $O(0, 0)$, $A\left(\frac{c}{m_1}, c\right)$, $B\left(\frac{c}{m_2}, c\right)$.

Take $AB$ as base:

Base length: $c * \left|\left(\frac{1}{m_1}\right) - \left(\frac{1}{m_2}\right)\right| = c * \left|\frac{m_2 - m_1}{m_1 * m_2}\right|$

Height $= c$.

Area: $\frac{1}{2} * c * c * \left|\frac{m_2 - m_1}{m_1 * m_2}\right| = \frac{c^2}{2} * \frac{|m_2 - m_1|}{|m_1 * m_2|}$

Now $m_1, m_2$ are roots of: $x^2 + \left(\sqrt{3} + 2\right)x + \left(\sqrt{3} - 1\right) = 0$

Sum: $m_1 + m_2 = -\left(\sqrt{3} + 2\right)$. Product: $m_1 * m_2 = \sqrt{3} - 1$

Difference: $|m_2 - m_1| = \sqrt{(m_1 + m_2)^2 - 4 * m_1 * m_2} = \sqrt{11}$

So area: $\Delta = \frac{c^2}{2} * \frac{\sqrt{11}}{\sqrt{3} - 1}$

196. Line perpendicular to $4x + 7y + 13 = 0$ is given by $7x - 4y + k = 0$. Since $P(-8, 12)$ lies on this perpendicular, therefore,

$k = 104$. So the equation of the perpendicular is $7x - 4y + 104 = 0$. This line intersects with the given line.

Solving both the equation we find the coordinate of the foot of the perpendicular as $(-12, 5)$.

197. Slope: $m = \frac{4-2}{5-(-1)} = \frac{2}{6} = \frac{1}{3}$. Equation using point $P$: $y - 2 = \left(\frac{1}{3}\right)(x + 1)$

$x - 3y + 7 = 0$

Let foot of perpendicular be $F(x, y)$.

Since $AF$ is perpendicular to $PQ$, slope of $PQ$ is $\frac{1}{3}$, so slope of $AF$ is $-3$.

Equation of $AF$ through $A(1, 0)$: $y - 0 = -3(x - 1) \Rightarrow y = -3x + 3$

Now solve with line $PQ$: $x - 3y + 7 = 0$

Substitute $y$: $x - 3(-3x + 3) + 7 = 0$

$x = \frac{1}{5}$, $y = -3\left(\frac{1}{5}\right) + 3 = -\frac{3}{5} + \frac{15}{5} = \frac{12}{5}$

198. Line perpendicular to $x + 3y = 3$ is $3x - y = k$ but it passes through origin so $k = 0$. Now intersection of $x + 3y = 3$ and $3x - y = 0$ is $\left(\frac{3}{10}, \frac{9}{10}\right)$.

Line perpendicular to $2x + 3y = 5$ is $3x - 2y = k'$ but it passes through origin so $k' = 0$. Now intersection of the two line is $\left(\frac{10}{13}, \frac{15}{13}\right)$.

Now line passing through these points is given by $33x - 61y + 45 = 0$.

199. Let $(h, r)$ be the foot of the perpendicular from $(x_1, y_1)$ to the line.

Since $(h, r)$ lies on the line: $lh + mr + n = 0 \quad \cdots(1)$

Slope of given line is $-\frac{l}{m}$. So slope of perpendicular is $\frac{m}{l}$.

Hence slope of line joining $(x_1, y_1)$ and $(h, r)$ is $\frac{m}{l}$:

$\frac{y_1 - r}{x_1 - h} = \frac{m}{l}$

So: $l(y_1 - r) = m(x_1 - h)$

$ly_1 - lr = mx_1 - mh \quad \cdots(2)$

Rearranging (2): $mh - lr = mx_1 - ly_1 \quad \cdots(3)$

Now multiplying (1) by $m$: $lmh + m^2 r + mn = 0 \quad \cdots(4)$

Multiplying (1) by $l$: $l^2 h + lmr + ln = 0 \quad \cdots(5)$

Now solving for $h$ and $r$ using (3),(4),(5).
$h = x_1 - \frac{l(lx_1 + my_1 + n)}{l^2 + m^2}$, $r = y_1 - \frac{m(lx_1 + my_1 + n)}{l^2 + m^2}$

Hence: $x_1 - h = \frac{l(lx_1 + my_1 + n)}{l^2 + m^2}$ $y_1 - r = \frac{m(lx_1 + my_1 + n)}{l^2 + m^2}$

So dividing: $\frac{x_1 - h}{l} = \frac{y_1 - r}{m} = \frac{lx_1 + my_1 + n}{l^2 + m^2}$.

200. Let image of $P$ be $P'(x, y)$. Let the foot of perpendicular be $F$ from $P$ to the line.

Slope of given line is $-\frac{4}{7}$, so perpendicular slope is $\frac{7}{4}$.

Equation of perpendicular through $(-8, 12)$: $y - 12 = \left(\frac{7}{4}\right)(x + 8)$

$4y - 48 = 7x + 56 \Rightarrow 7x - 4y + 104 = 0$

Now solving with $4x + 7y + 13 = 0$ give us the foot foot as $F(-12, 5)$.

Since $F$ is midpoint of $P(-8, 12)$ and $P'(x, y)$:

$x_F = \frac{x + (-8)}{2} = -12 \Rightarrow \frac{-8 + x}{2} = -12 \Rightarrow x - 8 = -24 \Rightarrow x = -16$

$y_F = \frac{y + 12}{2} = 5 \Rightarrow y + 12 = 10 \Rightarrow y = -2$.

201. Let $A(2, 1)$ and its image be $A'(5, 2)$. The mirror line is the perpendicular bisector of segment $AA'$.
Midpoint of $AA'$ is $M = \left(\frac{2+5}{2}, \frac{1+2}{2}\right) = \left(\frac{7}{2}, \frac{3}{2}\right)$

Slope of $AA'$ is $m = \frac{2-1}{5-2} = \frac{1}{3}$

So slope of mirror line is negative reciprocal is $-3$.

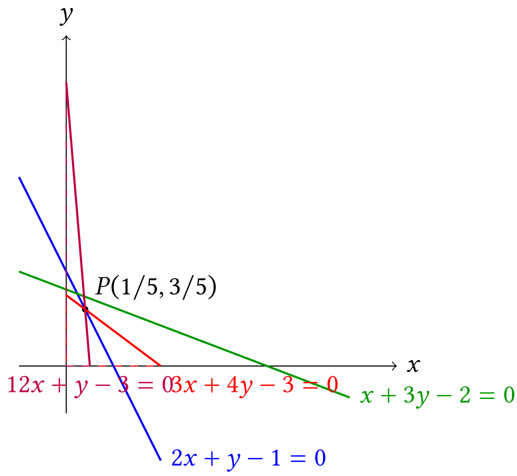
Equation of line through $M\left(\frac{7}{2}, \frac{3}{2}\right)$ with slope $-3$ is given by $y - \frac{3}{2} = -3\left(x - \frac{7}{2}\right) \Rightarrow 3x + y = +12$.

202. The point of intersection of the lines $3x + 2y = 0$ and $x - 2y = 0$ is $O(0, 0)$. Equation of the line passing through $O(0, 0)$ and $(1, 1)$ is given by $x = y$.

203. Given that $5x - y = 9 \Rightarrow y = 5x - 9$. Putting it in the line $x + 6y = 8 \Rightarrow x + 30x - 54 = 8 \Rightarrow x = 2 \Rightarrow y = 1$.

Equation of the line passing through $(2, 1)$ and $(2, -2)$ is given by $x = 2$.

204.




From first equation, $y = 1 - 2x \Rightarrow x + 3(1 - 2x) - 2 = 0 \Rightarrow x = \frac{1}{5}$

Then $y = 1 - 2\left(\frac{1}{5}\right) = \frac{3}{5}$. So intersection point is $P\left(\frac{1}{5}, \frac{3}{5}\right)$.

Let required line cut axes at $(a, 0)$ and $(0, b)$.

Equation in intercept form is given by $\frac{x}{a} + \frac{y}{b} = 1$

Since it passes through $P\left(\frac{1}{5}, \frac{3}{5}\right) \Rightarrow \frac{\frac{1}{5}}{a} + \frac{\frac{3}{5}}{b} = 1$

$\Rightarrow \frac{1}{a} + \frac{3}{b} = 5$

Area of triangle formed with axes is given by $\frac{1}{2} * a * b = \frac{3}{8} \Rightarrow ab = \frac{3}{4}$

$\Rightarrow \frac{b+3a}{a} b = 5 \Rightarrow \frac{b+3a}{\frac{3}{4}} = 5 \Rightarrow b + 3a = \frac{15}{4}$

$\Rightarrow b = \frac{15}{4} - 3a \Rightarrow a\left(\frac{15}{4} - 3a\right) = \frac{3}{4} \Rightarrow 4a^2 - 5a + 1 = 0$

$\Rightarrow a = 1$ or $a = \frac{1}{4}$

If $a = 1$, then $b = \frac{3}{4}$. If $a = \frac{1}{4}$, then $b = 3$.

Hence, equations are $x + \frac{y}{\frac{3}{4}} = 1 \Rightarrow 3x + 4y - 3 = 0$ and $\frac{x}{\frac{1}{4}} + \frac{y}{3} = 1 \Rightarrow 12x + y - 3 = 0$.

205. Let $AB$ is $2x - y + 1 = 0 \rightarrow y = 2x + 1$. Let $AD$ is $x + 3y - 10 = 0$

Substitute $y = 2x + 1$ into $AD$: $x + 3(2x + 1) - 10 = 0 \Rightarrow 7x - 7 = 0 \Rightarrow > x = 1, y = 3$

So $A = (1, 3)$ and $C = (-1, -2)$ Midpoint of $AC$: $M = \left(\frac{1+(-1)}{2}, \frac{3+(-2)}{2}\right) = \left(0, \frac{1}{2}\right)$.

So $M$ is midpoint of $BD$ also Let direction of $AB = (1, 2)$. Let direction of $AD = (3, -1)$.

So $B = (1,3) + t(1,2) = (1+t, 3+2t)$ and $D = (1,3) + s(3,-1) = (1+3s, 3-s)$

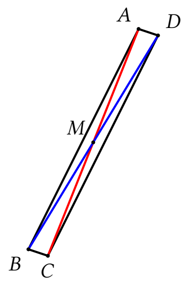
Midpoint condition: $((1+t) + \frac{1+3s}{2} = 0 \to t + 3s = -2 \Rightarrow \frac{(3+2t)+(3-s)}{2} = \frac{1}{2} \to 2t - s = -5$

$\Rightarrow t = -\frac{17}{7}, s = \frac{1}{7}$. So $B = \left(-\frac{10}{7}, -\frac{13}{7}\right)$ and $D = \left(\frac{10}{7}, \frac{20}{7}\right)$.

Slope of $AC$ is $\frac{-2-3}{-1-1} = \frac{5}{2}$ and slope of $BD$ is $\frac{\frac{20}{7}+\frac{13}{7}}{\frac{10}{7}+\frac{10}{7}} = \frac{33}{20}$

Equation of $AB$ is $y - 3 = \left(\frac{5}{2}\right)(x-1) \Rightarrow 5x - 2y + 1 = 0$

Equation of $BD$ is $33x - 20y + 10 = 0$.




206. From $2x - y - 5 = 0 \to y = 2x - 5$

Substituting into $3x - y - 6 = 0$ gives us $3x - (2x-5) - 6 = 0 \Rightarrow x = 1$

Now $y = 2 * 1 - 5 = -3$. So intersection point is $(1, -3)$.

$4x - y - 7 = 0$. Substitute $(1,-3)$ gives us $4(1) - (-3) - 7 = 4 + 3 - 7 = 0$

Since it satisfies the third equation, all three lines are concurrent at $(1, -3)$.

207. Since the point lies on the $y$-axis, $x = 0$.

Substituting $x = 0$ in first line gives us $(2m+3)y + m + 6 = 0 \to (2m+3)y = -(m+6) \Rightarrow y = -\frac{m+6}{2m+3}$

Substituting $x = 0$ in second line gives us $(m-1)y + m - 9 = 0 \to (m-1)y = 9 - m \Rightarrow y = \frac{9-m}{m-1}$

Equating both values of $y$ gives us $-\frac{m+6}{2m+3} = \frac{9-m}{m-1} \Rightarrow m = 21, -1$.

208. Substituting $y = x + 1$ into $2x + y = 16$ gives us $2x + (x+1) = 16$

Simplifying gives us $3x + 1 = 16$

Solving gives us $3x = 15$ so $x = 5$

Finding $y$ gives us $y = 5 + 1 = 6$

So the point of intersection being obtained is $(5, 6)$

Substituting $(5, 6)$ into $y = mx - 4$ gives us $6 = 5m - 4 \Rightarrow m = 2$.

209. Multiplying the first equation by $b$ is giving us $abx + a^2by + b = 0$

Multiplying the second equation by $a$ is giving us $abx + ab^2y + a = 0$

Subtracting gives us $a^2by - ab^2y + b - a = 0 \Rightarrow ab(a-b)y + (b-a) = 0$

$\Rightarrow (a-b)(aby-1) = 0$. So either $a = b$ or $aby = 1$.

If $a = b$, then at least two constants are already being equal.

If $aby = 1$, then $y = \frac{1}{ab}$.

Substituting $y = \frac{1}{ab}$ into $ax + a^2y + 1 = 0$ gives us $ax + a^2\left(\frac{1}{ab}\right) + 1 = 0$

Simplifying gives us $ax + \frac{a}{b} + 1 = 0$

So $x$ is $x = -\frac{\frac{a}{b}+1}{a}$

Substituting the same point into the third equation $cx + c^2y + 1 = 0$ gives us a condition relating $a, b, c$

After simplification it gives us $(a-c)(b-c) = 0$

So either $a = c$ or $b = c$. Therefore, at least two of $a, b, c$ are equal.

210. Equating $m_1x + c_1 = m_2x + c_2$ gives us $(m_1 - m_2)x = c_2 - c_1$

So $x$ is $x = \frac{c_2-c_1}{m_1-m_2}$. Finding $y$ gives us $y = \frac{m_1(c_2-c_1)}{m_1-m_2} + c_1$

The third line also passes through this so
$\frac{m_1(c_2-c_1)}{m_1-m_2} + c_1 = \frac{m_3(c_2-c_1)}{m_1-m_2} + c_3$

$\Rightarrow m_1(c_2-c_3) + m_2(c_3-c_1) + m_3(c_1-c_2) = 0.$

211. Multiplying $(b+c)x + ay + 1 = 0$ by $b$ gives us $b(b+c)x + aby + b = 0$

Multiplying $(c+a)x + by + 1 = 0$ by $a$ gives us $a(c+a)x + aby + a = 0$

Subtracting the equations gives us $b(b+c)x - a(c+a)x + b - a = 0$

$\Rightarrow (b-a)((a+b+c)x - 1) = 0$

So either $a = b$ or $(a+b+c)x = 1$.

If $a = b$, then two of the lines are already being identical in structure and concurrency is satisfied.

Assuming $a \neq b$, finding $x$ gives us $x = \frac{1}{a+b+c}$

Substituting $x = \frac{1}{a+b+c}$ into the first equation gives us $\frac{b+c}{a+b+c} + ay + 1 = 0$

Solving for $y$ yields $ay = -1 - \frac{b+c}{a+b+c}$

So $y = -\frac{a+b+c+b+c}{a(a+b+c)} = -\frac{a+2b+2c}{a(a+b+c)}$

Substituting this point into the third equation $(a+b)x + cy + 1 = 0$ yields $\frac{a+b}{a+b+c} + c\left(-\frac{a+2b+2c}{a(a+b+c)}\right) + 1 = 0$

Multiplying by $(a+b+c)$ yields $(a+b) - \frac{c(a+2b+2c)}{a} + (a+b+c) = 0$

Simplifying yields an identity equal to zero. So the same point is satisfying all three equations.

212. We consider a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

We take the perpendicular bisector of side $AB$. The midpoint of $AB$ is $M_1\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and the slope of $AB$ is $\frac{y_2-y_1}{x_2-x_1}$.

So the slope of its perpendicular bisector is $-\frac{x_2-x_1}{y_2-y_1}$.

Hence the equation of the perpendicular bisector of $AB$ is $\left(y - \frac{y_1+y_2}{2}\right) = -\frac{x_2-x_1}{y_2-y_1}\left(x - \frac{x_1+x_2}{2}\right)$

Similarly we form the perpendicular bisector of side $BC$. Its midpoint is $M_2\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$ and its equation is $\left(y - \frac{y_2+y_3}{2}\right) = -\frac{x_3-x_2}{y_3-y_2}\left(x - \frac{x_2+x_3}{2}\right)$

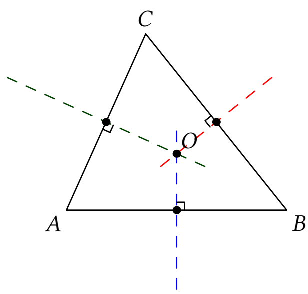
Now we solve these two equations simultaneously and obtain a point $(h, k)$.

This point $(h, k)$ satisfies both equations, so it lies on the perpendicular bisectors of $AB$ and $BC$. Hence it is equidistant from $A$ and $B$, and also from $B$ and $C$.

Thus we get $PA = PB$ and $PB = PC$ so we conclude $PA = PC$.

This shows that the point $(h, k)$ also lies on the perpendicular bisector of $AC$.

213.




Let $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ be the vertices of a $ABC$.

Let $P(x, y)$ lie on the perpendicular bisector of $AB$, then $PA = PB$

$\sqrt{(x-x_1)^2 + (y-y_1)^2} = \sqrt{(x-x_2)^2 + (y-y_2)^2}$

$\Rightarrow 2x(x_2 - x_1) + 2y(y_2 - y_1) = x_2^2 + y_2^2 - x_1^2 - y_1^2$

This is the perpendicular bisector of $AB$

Similarly for $BC$, $2x(x_3 - x_2) + 2y(y_3 - y_2) = x_3^2 + y_3^2 - x_2^2 - y_2^2$

The two linear equations intersect at a unique point and that point is equidistant from $A, B$, and $C$ iie. $PA = PB = PC$

Therefore, the perpendicular bisectors of a triangle are concurrent.

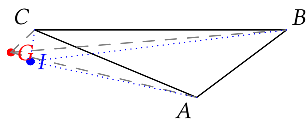
214. GIve equation is $x(1+\lambda) + y(2-\lambda) + 5 = 0$, which can be written as $x + 2y - 5 + \lambda(x - y) = 0$

The above equation represents two lines $x+2y-5=0$ and $x-y=0$ to be concurrent. Solving the two equations we find the fixed point as $\left(-\frac{5}{3}, -\frac{5}{3}\right)$.

215. $x(a+2b)+y(a-3b)=a-b$ can be rewritten as $a(x+y-1)+b(2x-3y+1)=0$, which represents two equations $x+y-1=$ and $2x-3y+1=0$, which are concurrent.

Solving the two equation yields the fixed point $\left(\frac{2}{5}, \frac{3}{5}\right)$, which is independent of $a$ and $b$.

216.



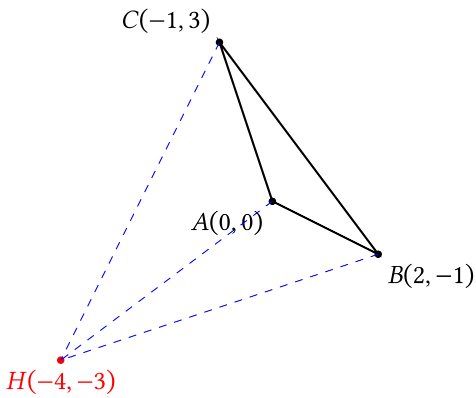

Solving first two equation gives $A\equiv(0,0)$, Solving first and last gives $B\equiv(20,15)$ and solving last two gives $C\equiv(-36,15)$

If $G$ be the centroid then $G\equiv\left(\frac{0+26-36}{3}, \frac{0+15+15}{3}\right)=\left(-\frac{16}{3}, 10\right)$.

$a=BC=39, b=CA=56, c=AB=39$

Using the formula for the incenter we have $I\equiv(-1,8)$.

217.




Let $A\equiv(0,0), B\equiv(2,-1)$ and $C\equiv(-1,3)$. Let $AL\perp BC$ then equation is $3x=4y$.

Let $BM\perp AC$, then equation of $BM$ is $x-3y=5$. Solving the equation of two perpendiculars we get orthocenter as $(-4,-3)$.

218. Consider a triangle $ABC$ with sides $AB: 3x-2y=6, BC: 3x+4y=-12$ and $AC: 3x-8y=-12$.

Solving first and last we get $A\equiv(4,3)$, and solving first two gives us $B\equiv(0,-3)$.

Let $AL \perp BC$ then equation of $AL$ is given by $4x - 3y - 7 = 0$, and if $BM \perp AC$ then equation of $BM$ is given by $8x + 3y + 9 = 0$.

Solving $AL$ and $BM$ gives us the orthocenter as $H \equiv \left(-\frac{1}{6}, -\frac{23}{9}\right)$.

219. Let $ABC$ be the given triangle and $B \equiv (3, -1)$ and $C \equiv (-2, 3)$. Let $H$ be the orthocenter of the $\triangle\, ABC$.

Given that $H \equiv (0, 0)$. Since $AB$ passes through $B(3, -1)$ and is perpendicular to the line $CH$ equation of $AB$ is $2x - 3y = 9$. Similarly equation of $AC$ is $3x - y = -9$.

Solving the two equations gives us $A \equiv \left(-\frac{36}{7}, -\frac{45}{7}\right)$.

220. Consider a $\triangle\, ABC$ such that $AB$ is $y = m_1 x$ and $AC$ is $y = m_2 x$. We also let equation of $BC$ as $lx + my = 1$.

Clearly, $A$ will be $(0, 0)$

Since $AH \perp BC \therefore \frac{b}{a\left(-\frac{l}{m}\right)} = -1 \Rightarrow \frac{l}{m} = \frac{m}{b} = k(\text{say})$

Coordinates of $B$ and $C$ are $\left(\frac{1}{l+mm_1}, \frac{m_1}{l+mm_1}\right)$ and $\left(\frac{1}{l+mm_2}, \frac{m_1}{l+mm_2}\right)$

Equation of perpendicular bisectors through $B$ and $C$ are

$x + m_2 y = \frac{1+m_1 m_2}{l+mm_1}$ and $x + m_1 y = \frac{1+m_1 m_2}{l+mm_2}$

Thus, coordinates of $H$ are $y = \frac{(1+m_1 m_2)m_1}{l^2+lm(m_1+m_2)+m^2 m_1 m_2}$

$m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$

Thus, $y = \frac{(a+b)m}{bl^2-2hlm_a m^2} \Rightarrow \frac{m}{b} = \frac{bl^2-2hlm+am^2}{a+b}$

$\Rightarrow k = \frac{a+b}{ab(a+b-2h)}$

Thus, equation of $BC$ is $lx + my = 1 \Rightarrow abx + bky = 1 \Rightarrow k(ax + by) = 1$

$\Rightarrow (ax + by)(a + b) = ab(a + b - 2h)$.

221. Consider the $\triangle\, ABC$ such that equation of $AB$ is $px + qy + r = 0$ and that of $AC$ is $lx + my + n = 0$.

Equation of any line passing through these lines is given by $px + qy + r + k(lx + my + n) = 0$. Slope of this line is $-\frac{p+kl}{q+km}$.

Equation of $BC$ is $-\frac{a}{b}$. Let $AD$ be the perpendicular through $A$ on $BC$, then $AD \perp BC$.

$\Rightarrow -\frac{p+kl}{q+km}.\left(-\frac{a}{b}\right) = -1 \Rightarrow k = \frac{cp+bq}{al+bm}$

Thus, equation of $AD$ is $\frac{px+qy+r}{ap+bq} = \frac{lx+my+n}{al+nb}$.

222. Let the equations of the sides $BC, CA$ and $AB$ of the $ABC$ are $L_1 \equiv = x\cos\theta_1 + y\sin\theta_1 - p_1 = 0, x\cos\theta_2 + y\sin\theta_2 - p_2 = 0$ and $x\cos\theta_3 + y\sin\theta_3 - p_3 = 0$.

Let $AD$ and $BE$ are perpendiculars through $A$ and $B$ on opposite sides.

Equation of $AD$ is given by $L_2 + kL_3 = 0 \Rightarrow x(\cos\theta_2 + \cos\theta_3) + y(\sin\theta_2 + \sin\theta_3) - (p_2 + kp_3) = 0$

Slope of $AD$ is $-\frac{\cos\theta_2+\cos\theta_3}{\sin\theta_2+\sin\theta_3} = m_1$

Slope of $BC$ is $-\frac{\cos\theta_1}{\sin\theta_1}$. Product of these two slopes would be $-1$.

$\Rightarrow k = -\frac{\cos(\theta_1-\theta_2)}{\cos(\theta_1-\theta_3)}$

Thus, equation of $AD$ becomes $L_2 - \frac{\cos(\theta_1-\theta_2)}{\cos(\theta_1-\theta_3)}L_3 = 0 \Rightarrow L_1\cos(\theta_1-\theta_3) = L_3\cos(\theta_1-\theta_2)$

And we proceed similarly for another perpendiculars to obtain the desired equation.

223. $A = (0,0)$ from $3x - 4y = 0$ and $5x + 12y = 0$

$B = (20,15)$ from $3x - 4y = 0$ and $y = 15$

$C = (-36,15)$ from $5x + 12y = 0$ and $y = 15$

Centroid is $G = \left(\frac{0+20-36}{3}, \frac{0+15+15}{3}\right) = \left(-\frac{16}{3}, 10\right)$

Length of the sides are, $a = |BC| = 56, b = |CA| = 39, c = |AB| = 25$

Incenter is $I = \left(\frac{ax_A+bx_B+cx_C}{a+b+c}, \frac{ay_A+by_B+cy_C}{a+b+c}\right) = \left(\frac{0+780-900}{120}, \frac{0+585+375}{120}\right) = (-1,8)$

224. Let $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$.

$AB^2 = (x_1-x_2)^2 + (y_1-y_2)^2$. $AC^2 = (x_1-x_3)^2 + (y_1-y_3)^2$, and $BC^2 = (x_2-x_3)^2 + (y_2-y_3)^2$

Consider $AB^2 + AC^2 - BC^2 = \left[(x_1-x_2)^2 + (y_1-y_2)^2\right] + \left[(x_1-x_3)^2 + (y_1-y_3)^2\right] - \left[(x_2-x_3)^2 + (y_2-y_3)^2\right]$

$= (x_1^2 - 2x_1x_2 + x_2^2) + (y_1^2 - 2y_1y_2 + y_2^2) + (x_1^2 - 2x_1x_3 + x_3^2) + (y_1^2 - 2y_1y_3 + y_3^2) - (x_2^2 - 2x_2x_3 + x_3^2) - (y_2^2 - 2y_2y_3 + y_3^2)$

$= 2[(x_1^2 - x_1x_2 - x_1x_3 + x_2x_3) + (y_1^2 - y_1y_2 - y_1y_3 + y_2y_3)]$

$= 2[(x_1-x_2)(x_1-x_3) + (y_1-y_2)(y_1-y_3)]$

Thus, $AB^2 + AC^2 - BC^2 = 2[(x_1-x_2)(x_1-x_3) + (y_1-y_2)(y_1-y_3)]$

We know that if $\angle A$ is acute if $AB^2 + AC^2 - BC^2 > 0$ and obtuse if $< 0$ and for right angle should be equal to zero.

225. Given lines are $L_1 : 4x - 3y = 5$, $L_2 : x - 2y = 10$, $L_3 : 7x + y = 40$, and $L_4 : x + 3y + 10 = 0$

Let $A$ be the intersection of $L_1$ and $L_2$, $B$ be the intersection of $L_2$ and $L_3$, $C$ be the intersection of $L_3$ and $L_4$, and $D$ be the intersection of $L_4$ and $L_1$

Intersection of $L_1$ and $L_2$) is $4x - 3y = 5$ and $x - 2y = 10$

From $x - 2y = 10$, we get $x = 10 + 2y \Rightarrow 4(10+2y) - 3y = 5 \Rightarrow 5y = -35 \Rightarrow y = -7 \Rightarrow x = 10 + 2(-7) = -4 \Rightarrow A$ is $(-4,-7)$

Similarly we find that other points are $B(6,-2)$, $C\left(\frac{13}{2},-\frac{11}{2}\right)$, and $D(-1,-3)$

We find the slopes as $m_{\{AB\}} = \frac{-2+7}{6+4} = \frac{1}{2}$, $m_{\{BC\}} = \frac{\frac{-11}{2}+2}{\frac{13}{2}-6} = -7$, $m_{\{CD\}} = (-3 + \frac{\frac{11}{2}}{-1-\frac{13}{2}} = -\frac{1}{3}$, and $m_{\{AA\}} = \frac{-7+3}{-4+1} = \frac{4}{3}$

Now we check angle relations using: $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

We find that $\angle A + \angle C = 180^{\circ}$. Hence, the quadrilateral is cyclic.

226. Let the four sides of the quadrilateral taken in order be $L_1 : a_1x + b_1y + c_1 = 0$, $L_2 : a_2x + b_2y + c_2 = 0$, $L_3 : a_3x + b_3y + c_3 = 0$, and $L_4 : a_4x + b_4y + c_4 = 0$.

Let the vertices be $A = L_1 \cap L_2$, $B = L_2 \cap L_3$, $C = L_3 \cap L_4$, $D = L_4 \cap L_1$

A quadrilateral is cyclic iff opposite angles are supplementary i.e. $\angle A + \angle C = 180°$

For a line $ax + by + c = 0$, slope is $m = -\frac{a}{b}$

Angle between two lines $L_1$ and $L_2$ isa $\tan\theta = \left|\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\right|$

$\Rightarrow \tan A = \left|\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}\right|$

$\Rightarrow \tan C = \left|\frac{a_4b_3-a_3b_4}{a_3a_4+b_3b_4}\right|$

For cyclic quadrilateral, $\angle A + \angle C = 180°$

$\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2} = \frac{a_4b_3-a_3b_4}{a_3a_4+b_3b_4}$

$\Rightarrow (a_2b_1 - a_1b_2)(a_3a_4 + b_3b_4) = (a_4b_3 - a_3b_4)(a_1a_2 + b_1b_2)$.

227. Show that the lines $2x + 3y + 19 = 0$ and $9x + 6y - 17 = 0$ cut the coordinate axes in concyclic points.

Let the given lines be $L_1 : 2x + 3y + 19 = 0$, and $L_2 : 9x + 6y - 17 = 0$.

For $L_1$, x-intercept: put $y = 0 \Rightarrow 2x + 19 = 0 \Rightarrow x = -\frac{19}{2}$. So $A$ is $\left(-\frac{19}{2}, 0\right)$

y-intercept: put $x = 0 \Rightarrow 3y + 19 = 0 \Rightarrow y = -\frac{19}{3}$. So $B$ is $\left(0, -\frac{19}{3}\right)$

Similarly we find $C$ and $D$ for $L_2$ as $C\left(\frac{17}{9}, 0\right)$ $D\left(0, \frac{17}{6}\right)$

Now we check concyclicity using the condition that four points are concyclic if the angle subtended by the same chord is equal.

Consider chord $AC$ on the x-axis.

Slope of $AB$ is $m_{\{AB\}} = \frac{-\frac{19}{3}-0}{0+\frac{19}{2}} = -\frac{2}{3}$

Slope of $BC$ is $m_{\{BC\}} = \frac{-\frac{19}{3}-0}{0-\frac{17}{9}} = \frac{57}{17}$

So angle at $B$ is angle between lines with slopes $-\frac{2}{3}$ and $\frac{57}{17}$.

Now slope of $AD$ is $m_{\{AD\}} = \frac{\frac{17}{6}-0}{0+\frac{19}{2}} = \frac{\frac{17}{6}}{\frac{19}{2}} = \frac{17}{57}$

Slope of $CD$ is $m_{\{CD\}} = \frac{\frac{17}{6}-0}{0-\frac{17}{9}} = \frac{\frac{17}{6}}{-\frac{17}{9}} = -\frac{3}{2}$

So angle at $D$ is angle between slopes $\frac{17}{57}$ and $-\frac{3}{2}$.

Now compute angle between two lines using: $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$

For angle at $B$ is $\tan\angle B = \left|\frac{\frac{57}{17} + \frac{2}{3}}{1 - \left(\frac{2}{3}\right)\left(\frac{57}{17}\right)}\right| = 1$

For angle at $D$ is $\tan\angle D = \left|\frac{-\frac{3}{2} - \frac{17}{57}}{1 - \left(\frac{17}{57}\right)\left(-\frac{3}{2}\right)}\right| = 1$

Thus $\angle B = \angle D$. Since equal angles subtend the same chord, the four points are concyclic.

228. Let $B(-4, -5)$ be a vertex of triangle $ABC$.

$5x + 3y - 4 = 0$ is altitude from $A$, so slope of $BC$ is $\frac{3}{5}$. Equation of $BC$ through $B(-4, -5)$ is $y + 5 = \left(\frac{3}{5}\right)(x + 4) \Rightarrow 3x - 5y - 13 = 0$

$3x + 8y + 13 = 0$ is altitude from $C$, so slope of $AB$ is $\frac{8}{3}$.

Equation of $AB$ through $B(-4, -5)$ is $y + 5 = \left(\frac{8}{3}\right)(x + 4) \Rightarrow 8x - 3y + 17 = 0$

We find $A$ as intersection of $AB$ and altitude $5x + 3y - 4 = 0$

$13x + 13 = 0 \Rightarrow x = -1 \Rightarrow 5(-1) + 3y - 4 = 0 \Rightarrow y = 3$. So A is $(-1, 3)$.

Similarly $C$ is $(1, -2)$

Equation of $AC$ is $5x + 2y - 1 = 0$

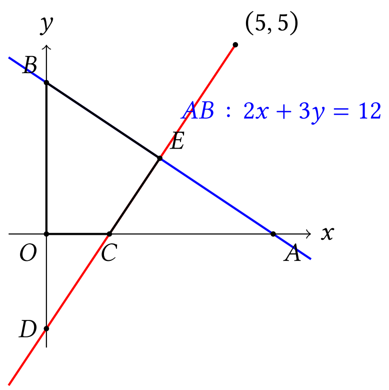
So the sides are: $AB : 8x - 3y + 17 = 0$, $BC : 3x - 5y - 13 = 0$, and $AC : 5x + 2y - 1 = 0$.

229. Equation of the line perpendiculars to $5x - y = 1$ is given by $x + 5y = k$. This will make an intercept of $k$ with $x$-axis, and an intercept of $\frac{k}{5}$ with $y$-axis.

Thus, area of triangle, which is given as 5, is $\frac{1}{2}.k.\frac{k}{5} = 25 \Rightarrow k = \pm 5\sqrt{2}$.

So the equation of the line is $x + 5y = \pm 5\sqrt{2}$.

230.




Clearly, $A = (6, 0)$ and $B = (0, 4)$ as given line is $\frac{x}{6} + \frac{y}{4} = 1$.

We rewrite the line as $y = -\frac{2}{3}x + 4$, so the slope of $AB$ is $-\frac{2}{3}$.

A perpendicular line has slope $\frac{3}{2}$, so we form the line through $(5,5)$ as $y - 5 = \frac{3}{2}(x - 5) \Rightarrow y = \frac{3}{2}x - \frac{5}{2}$.

Now we find point $C$ by setting $y = 0$: $0 = \frac{3}{2}x - \frac{5}{2}$, so $x = \frac{5}{3}$. Thus $C = \left(\frac{5}{3}, 0\right)$.

We find point $D$ by setting $x = 0$: $y = -\frac{5}{2}$, so $D = \left(0, -\frac{5}{2}\right)$.

Now we find point $E$ by solving intersection of $2x + 3y = 12$ and $y = \frac{3}{2}x - \frac{5}{2}$.

Thus $x = 3$, and $y = 2$. So $E = (3, 2)$.

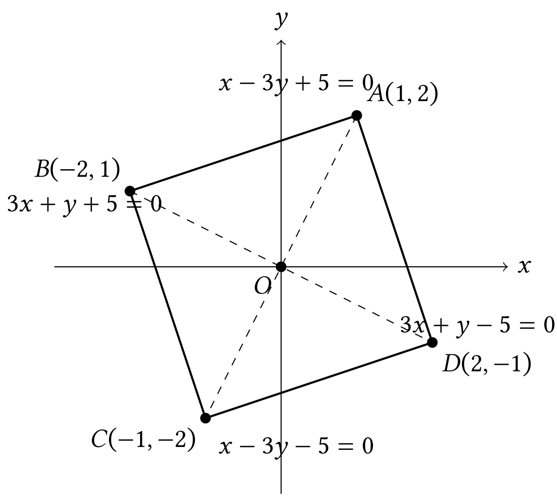
We compute the area by splitting the quadrilateral $OCEB$ into two triangles $OCE$ and $OEB$.

We first compute the area of triangle $OCE$. Since $O = (0, 0)$, we use the determinant shortcut: $\Delta OCE = \frac{1}{2} \mid x_C y_E - x_E y_C \mid = \frac{5}{3}$.

$\Delta OEB = \frac{1}{2} \mid x_E y_B - x_B y_E \mid \Rightarrow \frac{1}{2} \mid 3 * 4 - 0 * 2 \mid = 6.$

$\square OCEB = \frac{5}{3} + 6 = \frac{23}{3}$.

231.




Since the square is centered at the origin, we generate the remaining vertices by rotating the point $(1, 2)$ by $90^{\circ}$ repeatedly about the origin.

We rotate $(x, y)$ counterclockwise by $90^{\circ}$ using the rule $(x, y) \to (-y, x)$.

So we obtain $A = (1, 2)$, $B = (-2, 1)$, $C = (-1, -2)$, and $D = (2, -1)$.

We first take line $AB$. We compute its slope as $\frac{1-2}{-2-1} = \frac{1}{3}$.

So we write $y - 2 = \frac{1}{3}(x - 1)$ and simplify it to $x - 3y + 5 = 0$.

Similarly we find remaining sides to be $3x + y + 5 = 0, x - 3y - 5 = 0$, and $3x + y - 5 = 0$.

232. We first find the slope of $AD$ as $\frac{6-2}{-2-1} = \frac{4}{-3} = -\frac{4}{3}$.

Since $BC$is perpendicular to $AD$, we take the slope of $BC$as $\frac{3}{4}$.

So the line $BC$passes through $D(-2, 6)$and we write $y - 6 = \frac{3}{4}(x + 2)$, which simplifies to $4y - 24 = 3x + 6$, hence $3x - 4y + 30 = 0$.

We note that the altitude in an equilateral triangle satisfies $h = \left(\frac{\sqrt{3}}{2}\right)s$.

We compute the altitude as $AD = \sqrt{(1+2)^2 + (2-6)^2} = \sqrt{9+16} = 5$.

So the side length is $s = 2\frac{h}{\sqrt{3}} = \frac{10}{\sqrt{3}}$.

Since $D$lies on $BC$, we take direction vector of $BC$ as $(4, 3)$with magnitude 5, so the unit direction is $\left(\frac{4}{5}, \frac{3}{5}\right)$.

Half the side length is $\frac{s}{2} = \frac{5}{\sqrt{3}}$.

So we move from $D$to $B$and $C$using this direction $\left(\frac{5}{\sqrt{3}}\right) * \left(\frac{4}{5}, \frac{3}{5}\right) = \left(\frac{4}{\sqrt{3}}, \frac{3}{\sqrt{3}}\right)$.

Therefore, $B = \left(-2 - \frac{4}{\sqrt{3}}, 6 - \frac{3}{\sqrt{3}}\right)$ and $C = \left(-2 + \frac{4}{\sqrt{3}}, 6 + \frac{3}{\sqrt{3}}\right)$.

Now we find line $AB$using points $A(1, 2)$and $B$ i.e. $y - 2 = m_{AB}(x - 1)$ where $m_{AB} = \frac{6 - \frac{3}{\sqrt{3}} - 2}{-2 - \frac{4}{\sqrt{3}} - 1}$.

Similarly we can find other sides.

233. We have one side $x - y = 0$ so its slope is 1. In an equilateral triangle, the angle between sides is $60^{\circ}$. Hence slopes of the other sides satisfy

$m = \tan(45^{\circ} \pm 60^{\circ})$

So $m = \tan(105^{\circ}) = -2 - \sqrt{3}$ or $m = \tan(-15^{\circ}) = 2 - \sqrt{3}$.

Using the given vertex $\left(2 + \sqrt{3}, 5\right)$ and slope $2 - \sqrt{3}$

$y - 5 = \left(2 - \sqrt{3}\right)\left(x - \left(2 + \sqrt{3}\right)\right)$

Simplifying $y = \left(2 - \sqrt{3}\right)x + 6$

So second side is $y + \left(2 - \sqrt{3}\right)x = 6$.

For the third side take slope $-2 - \sqrt{3} \Rightarrow y - 5 = \left(-2 - \sqrt{3}\right)\left(x - \left(2 + \sqrt{3}\right)\right)$

$y + \left(2 + \sqrt{3}\right)x = 12 + 4\sqrt{3}$.

234. We have diagonal $8x - 15y = 0$ so its slope is $\frac{8}{15}$. Hence the other diagonal has slope $-\frac{15}{8}$.

In a square, diagonals bisect at right angles. Let the center be $(h, k)$ on $8h - 15k = 0$.

Since $(1, 2)$ is a vertex, the midpoint lies on the line through $(1, 2)$ with slope $-\frac{15}{8}$

$k - 2 = \left(-\frac{15}{8}\right)(h - 1)$

Solving with $8h-15k=0$ gives $(h,k)=\left(\frac{16}{17},\frac{120}{17}\right)$.

Slope of side is perpendicular to diagonal slope $\frac{8}{15}$, so slope of side is $-\frac{15}{8}$ or $\frac{8}{15}$ rotated by $45^{\circ}$. Thus, side slopes are $\frac{1}{8}$ and $-8$.

Through $(1,2) \Rightarrow y-2=\left(\frac{1}{8}\right)(x-1)$ gives $x-8y+15=0$ and $y-2=-8(x-1)$ gives $8x+y-10=0$.

235. $5y=12x+6$ gives slope $\frac{12}{5}$. $3x=4y+7$ gives slope $\frac{3}{4}$

Let required line have slope $m$. For equal angles
$\frac{m-\frac{12}{5}}{1+\frac{12}{5}m}=-\frac{m-\frac{3}{4}}{1+\frac{3}{4}m}$

Solving gives $m=1$ or $m=-1$.

Since these lines pass through $(4,5)$, thereforem $y-5=1(x-4)$ gives $y=x+1$ and $y-5=-1(x-4)$ gives $y=-x+9$.

236. The two given lines are $7x-y+3=0$ and $x+y-3=0$. Solving them together gives $y=7x+3$ and $y=3-x$, so $7x+3=3-x$, which gives $x=0$ and $y=3$. Thus the vertex of the triangle is $(0,3)$.

The angle bisectors are found from $\frac{7x-y+3}{5}=\pm(x+y-3)$. Solving gives the two bisectors $x-3y+9=0$ and $3x+y-3=0$.

The internal bisector is $3x+y-3=0$, whose slope is $-3$. The base is perpendicular to this bisector, so its slope is $\frac{1}{3}$.

Since the base passes through $(1,-10)$, its equation is $y+10=\left(\frac{1}{3}\right)(x-1)$. Simplifying gives $3y+30=x-1$, hence the required equation is $x-3y-31=0$.

237. The three lines are $x\cos\alpha+y\sin\alpha=p_1$, $x\cos\beta+y\sin\beta=p_2$, and $x\cos\gamma+y\sin\gamma=p_3$.

The intersection point of the first two lines is obtained by solving $x\cos\alpha+y\sin\alpha=p_1$ and $x\cos\beta+y\sin\beta=p_2$. Using determinants, this gives $x=\frac{p_1\sin\beta-p_2\sin\alpha}{\sin(\beta-\alpha)}$ and $y=\frac{p_2\cos\alpha-p_1\cos\beta}{\sin(\beta-\alpha)}$.

Similarly, the other two vertices are obtained by cyclic permutation of $\alpha,\beta,\gamma$ and $p_1,p_2,p_3$.

The area of the triangle formed by three lines is given by $\Delta=\frac{1}{2}|x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)|$.

Substituting the coordinates of the three intersection points and simplifying using the identity $\sin(A-B)=\sin A\cos B-\cos A\sin B$, the expression is reduced.

The numerator becomes $(p_1\sin(\gamma-\beta)+p_2\sin(\alpha-\gamma)+p_3\sin(\beta-\alpha))^2$ and the denominator becomes $|\sin(\gamma-\beta)\sin(\alpha-\gamma)\sin(\beta-\alpha)|$.

Thus the area is $\frac{1}{2}\frac{(p_1\sin(\gamma-\beta)+p_2\sin(\alpha-\gamma)+p_3\sin(\beta-\alpha))^2}{|\sin(\gamma-\beta)\sin(\alpha-\gamma)\sin(\beta-\alpha)|}$.

238. The line $L$ passes through $(1,1)$ and $(2,0)$, so its slope is $m = \frac{0-1}{2-1} = -1$. Hence its equation is $y - 1 = -1(x-1)$, which simplifies to $y = -x + 2$.

A line perpendicular to $L$ has slope 1. Passing through $\left(\frac{1}{2}, 0\right)$, its equation is $y - 0 = 1\left(x - \frac{1}{2}\right)$, so $y = x - \frac{1}{2}$.

The three lines are $x = 0$, $y = -x + 2$, and $y = x - \frac{1}{2}$.

The intersection of $x = 0$ and $y = -x + 2$ is $(0,2)$. The intersection of $x = 0$ and $y = x - \frac{1}{2}$ is $\left(0, -\frac{1}{2}\right)$.

The intersection of $y = -x + 2$ and $y = x - \frac{1}{2}$ is found by solving $-x + 2 = x - \frac{1}{2}$, which gives $2x = \frac{5}{2}$, so $x = \frac{5}{4}$ and $y = \frac{3}{4}$.

Thus the vertices of the triangle are $(0,2)$, $\left(0, -\frac{1}{2}\right)$, and $\left(\frac{5}{4}, \frac{3}{4}\right)$.

The base along the $y$-axis has length $\left|2 - \left(-\frac{1}{2}\right)\right| = \frac{5}{2}$, and the perpendicular distance of $\left(\frac{5}{4}, \frac{3}{4}\right)$ from the $y$-axis is $\frac{5}{4}$.

Hence the area is $\left(\frac{1}{2}\right) * \left(\frac{5}{2}\right) * \left(\frac{5}{4}\right) = \frac{25}{16}$.

239. The vertices of the triangle are $A(0,0)$, $B(8,0)$, and $C(4,8)$, and the given point is $P(9,3)$.

The side $AB$ has equation $y = 0$. The foot of the perpendicular from $P(9,3)$ to this line is clearly $(9,0)$.

The side $BC$ passes through $(8,0)$ and $(4,8)$, so its slope is $m = \frac{8-0}{4-8} = -2$. Hence its equation is $y = -2x + 16$, or $2x + y - 16 = 0$.

The foot of the perpendicular from $(9,3)$ to $2x + y - 16 = 0$ is given by $x = 9 - 2\frac{2*9+3-16}{2^2+1^2}$ and $y = 3 - \frac{2*9+3-16}{2^2+1^2}$.

Since $2 * 9 + 3 - 16 = 5$, this gives $x = 9 - \frac{10}{5} = 7$ and $y = 3 - \frac{5}{5} = 2$, so the foot is $(7,2)$.

The side $CA$ passes through $(4,8)$ and $(0,0)$, so its slope is 2. Its equation is $y = 2x$, or $2x - y = 0$.

The foot of the perpendicular from $(9,3)$ to $2x - y = 0$ is $x = 9 - 2\frac{2*9-3}{2^2+(-1)^2}$ and $y = 3 - (-1)\frac{2*9-3}{2^2+(-1)^2}$.

Since $29 - 3 = 15$, this gives $x = 9 - \frac{30}{5} = 3$ and $y = 3 + \frac{15}{5} = 6$, so the foot is $(3,6)$.

Thus the three feet are $(9,0)$, $(7,2)$, and $(3,6)$.

To check collinearity, the slope between $(9,0)$ and $(7,2)$ is $\frac{2-0}{7-9} = -1$, and the slope between $(7,2)$ and $(3,6)$ is $\frac{6-2}{3-7} = -1$, so the points lie on a straight line.

The equation of the line through $(9,0)$ with slope $-1$ is $y - 0 = -1(x - 9)$, which simplifies to $x + y - 9 = 0$.

240. The line is $\frac{x}{a} + \frac{y}{b} = 1$, which can be written as $bx + ay - ab = 0$.

The foot of the perpendicular from the origin $(0, 0)$ to the line $bx + ay - ab = 0$ is given by $\alpha = -\frac{b(-ab)}{a^2+b^2}$ and $\beta = -\frac{a(-ab)}{a^2+b^2}$.

Thus $\alpha = a\frac{b^2}{a^2+b^2}$ and $\beta = a^2\frac{b}{a^2+b^2}$.

Now $\alpha^2 + \beta^2 = \frac{a^2b^4+a^4b^2}{(a^2+b^2)^2} = a^2b^2\frac{a^2+b^2}{(a^2+b^2)^2} = a^2\frac{b^2}{a^2+b^2}$.

Also $\alpha + \beta = \frac{ab^2+a^2b}{a^2+b^2} = a\frac{b(a+b)}{a^2+b^2}$.

Thus $(\alpha^2 + \beta^2)(\alpha + \beta) = \left(a^2\frac{b^2}{a^2+b^2}\right) * \left(a\frac{b(a+b)}{a^2+b^2}\right) = a^3b^3\frac{a+b}{(a^2+b^2)^2}$.

Further $\alpha\beta = \left(a\frac{b^2}{a^2+b^2}\right) * \left(a^2\frac{b}{a^2+b^2}\right) = a^3\frac{b^3}{(a^2+b^2)^2}$.

Hence $(a + b)\alpha\beta = (a + b)\left(a^3\frac{b^3}{(a^2+b^2)^2}\right) = a^3b^3\frac{a+b}{(a^2+b^2)^2}$.

Thus $(\alpha^2 + \beta^2)(\alpha + \beta) = (a + b)\alpha\beta$.

241. The given lines are $x = 0$, $y = 0$, $x + y = 1$, and $6x + y = 3$.

The vertices of the quadrilateral are obtained by pairwise intersections. The origin $(0, 0)$ is one vertex. The intersection of $y = 0$ and $x + y = 1$ is $(1, 0)$. The intersection of $x + y = 1$ and $6x + y = 3$ is found by subtracting the equations, giving $5x = 2$, so $x = \frac{2}{5}$ and $y = \frac{3}{5}$. The intersection of $6x + y = 3$ and $x = 0$ is $(0, 3)$.

Thus the quadrilateral has vertices $(0, 0)$, $(1, 0)$, $\left(\frac{2}{5}, \frac{3}{5}\right)$, and $(0, 3)$.

The diagonal through the origin connects $(0, 0)$ and the opposite vertex $\left(\frac{2}{5}, \frac{3}{5}\right)$.

The slope of this diagonal is $\frac{\frac{3}{5}}{\frac{2}{5}} = \frac{3}{2}$, so its equation is $y = \left(\frac{3}{2}\right)x$, or $3x - 2y = 0$.

242. The given points are $A(-4, 5)$, $D\left(\frac{16}{5}, -\frac{23}{5}\right)$, $E(4, 1)$, and $F(-1, -4)$. The points $D, E, F$ are the feet of the perpendiculars from $A, B, C$ respectively.

Since $AD$ is perpendicular to $BC$, the slope of $BC$ is the negative reciprocal of the slope of $AD$. The slope of $AD$ is $\frac{\left(-\frac{23}{5}\right)-5}{\left(\frac{16}{5}\right)-(-4)} = \frac{\frac{-48}{5}}{\frac{36}{5}} = -\frac{4}{3}$, so the slope of $BC$ is $\frac{3}{4}$.

Thus the equation of $BC$ passing through $D\left(\frac{16}{5}, -\frac{23}{5}\right)$ is $y + \frac{23}{5} = \left(\frac{3}{4}\right)\left(x - \frac{16}{5}\right)$.

Now $E(4, 1)$ lies on $AC$ and $BE$ is perpendicular to $AC$. The slope of $AC$ is $\frac{1-5}{4-(-4)} = -\frac{4}{8} = -\frac{1}{2}$, so the slope of $BE$ is 2.

Thus the equation of $BE$ is $y - 1 = 2(x - 4)$, or $y = 2x - 7$.

The point $B$ lies on both $BC$ and $BE$. Solving $y = 2x - 7$ and $y + \frac{23}{5} = \left(\frac{3}{4}\right)\left(x - \frac{16}{5}\right)$ gives $x = 3$ and $y = -1$. Hence $B(3, -1)$.

Now $F(-1, -4)$ lies on $AB$ and $CF$ is perpendicular to $AB$. The slope of $AB$ is $\frac{-1-5}{3-(-4)} = -\frac{6}{7}$, so the slope of $CF$ is $\frac{7}{6}$.

Thus the equation of $CF$ is $y+4=\left(\frac{7}{6}\right)(x+1)$.

The point $C$ lies on both $BC$ and $CF$. Solving $y+\frac{23}{5}=\left(\frac{3}{4}\right)\left(x-\frac{16}{5}\right)$ and $y+4=\left(\frac{7}{6}\right)(x+1)$ gives $x=6$ and $y=3$. Hence $C(6,3)$.

Thus the required vertices are $B(3,-1)$ and $C(6,3)$.

243. The lines are $y=m_rx+c_r$ for $r=1,2,3$, and the transversal is $x+y=1$.

The point of intersection of the line $y=m_rx+c_r$ with $x+y=1$ is obtained by substituting $y=1-x$ into the line, giving $1-x=m_rx+c_r$. This simplifies to $x(1+m_r)=1-c_r$, so $x_r=\frac{1-c_r}{1+m_r}$ and $y_r=1-x_r$.

Thus the three points of intersection correspond to parameters $x_1,x_2,x_3$ on the line $x+y=1$.

The intercept cut off between two such points along the transversal is proportional to the difference of their $x$-coordinates, since all points lie on the same straight line.

Hence equal intercepts imply $x_2-x_1=x_3-x_2$, so $2x_2=x_1+x_3$.

Substituting $x_r=\frac{1-c_r}{1+m_r}$, this gives $2\frac{1-c_2}{1+m_2}=\frac{1-c_1}{1+m_1}+\frac{1-c_3}{1+m_3}$.

For the intercepts to be equal for arbitrary $c_r$, this condition reduces to $\frac{2}{1+m_2}=\frac{1}{1+m_1}+\frac{1}{1+m_3}$, which is needed condition.

244. The given pair of lines are $5x-y+4=0$ and $3x+4y-4=0$.

Let the required line cut these two lines at $A$ and $B$ respectively, and let the midpoint of $AB$ be $(1,5)$.

We know that that if a line through midpoint $(x_0,y_0)$ joins intersections with two lines $L_1=0$ and $L_2=0$, then its equation is $L_1+kL_2=0$ for some constant $k$.

So the required line is $5x-y+4+k(3x+4y-4)=0$ which simplifies to $(5+3k)x+(-1+4k)y+(4-4k)=0$.

Since $(1,5)$ lies on the line, we substitute $x=1$, $y=5$ to get $(5+3k)+5(-1+4k)+(4-4k)=0$.

This gives $5+3k-5+20k+4-4k=0$, so $4+19k=0$, hence $k=-\frac{4}{19}$.

Substituting back, the equation becomes $5x-y+4-\left(\frac{4}{19}\right)(3x+4y-4)=0$.

Multiplying by 19 gives $95x-19y+76-(12x+16y-16)=0$.

So $83x-35y+92=0$ is the required line.

245. The line $a_1x+b_1y+c_1=0$ cuts the axes at $A_1\left(-\frac{c_1}{a_1},0\right)$ and $B_1\left(0,-\frac{c_1}{b_1}\right)$. Hence the intercept form is $\frac{x}{-\frac{c_1}{a_1}}+\frac{y}{-\frac{c_1}{b_1}}=1$.

So the ratio of intercepts on the axes is $A_1=-\frac{c_1}{a_1}$ on the $x$-axis and $B_1=-\frac{c_1}{b_1}$ on the $y$-axis. Thus $\frac{A_1}{B_1}=\frac{b_1}{a_1}$.

Similarly, for $a_2x+b_2y+c_2=0$, the intercept ratio is $\frac{A_2}{B_2}=\frac{b_2}{a_2}$.

Since the lines cut the coordinate axes in cyclic points, the intercepts are in cyclic order, which implies the ratios of corresponding segments satisfy $\frac{A_1}{B_1} = \frac{B_2}{A_2}$.

Hence $\frac{b_1}{a_1} = \frac{a_2}{b_2}$. Taking absolute values yields $|a_1 a_2| = |b_1 b_2|$.

246. The rectangle $ABCD$ is inscribed in a circle, so its diagonals are diameters of the circle.

The given line $3y = x + 10$, i.e. $y = \frac{x+10}{3}$, is therefore the line containing one diagonal of the rectangle, so either $AC$ or $BD$ lies on this line.

The points are $A(-6, 7)$ and $B(4, 7)$, so $AB$ is horizontal since both points have the same $y$-coordinate. Hence $AB$ is a side of the rectangle and the adjacent side is vertical, so the rectangle is axis-aligned.

Thus $C$ lies vertically above $B$ and $D$ lies vertically above $A$, so we take $C(4, t)$ and $D(-6, t)$ for some $t$.

The diagonal $AC$ lies on the line $y = \frac{x+10}{3}$, so $A(-6, 7)$ and $C(4, t)$ satisfy the equation of this line.

Substituting $A$ is consistent since $7 = \frac{-6+10}{3} = \frac{4}{3}$ is false, so $A$ and $C$ cannot both lie on that line. Hence the diagonal is $BD$ instead.

So $B(4, 7)$ and $D(-6, t)$ lie on $y = \frac{x+10}{3}$.

For $B(4, 7)$, the line gives $7 = \frac{4+10}{3} = \frac{14}{3}$, which is false, so $B$ is not on that diagonal either. Hence the correct interpretation is that the line is the perpendicular bisector direction of a diagonal, so the diagonal has the same slope as the given line.

Thus, slope of diagonal $AC$ is $\frac{1}{3}$.

Now $A(-6, 7)$ and $C(4, t)$ lie on a line of slope $\frac{1}{3}$, so $\frac{t-7}{4-(-6)} = \frac{1}{3}$, giving $\frac{t-7}{10} = \frac{1}{3}$, hence $t - 7 = \frac{10}{3}$ and $t = \frac{31}{3}$.

So $C$ is $\left(4, \frac{31}{3}\right)$ and $D$ is $\left(-6, \frac{31}{3}\right)$.

The base $AB = 4 - (-6) = 10$ and the height is $\frac{31}{3} - 7 = \frac{31}{3} - \frac{21}{3} = \frac{10}{3}$.

Hence, the area is $10 * \left(\frac{10}{3}\right) = \frac{100}{3}$.

247. From the point $(2, 5)$, rays are drawn making an angle of $45^{\circ}$ with the line $2x + y = 1$.

Slope of the given line: $m_1 = -2$ Using the angle formula: $\tan 45^{\circ} = \left|\frac{m-(-2)}{1+m*(-2)}\right| = 1$

So, $\frac{m+2}{1-2m} = \pm 1$

Solving $m = -\frac{1}{3}$ and $m = 3$

Equations of incident rays: $y - 5 = \left(-\frac{1}{3}\right)(x - 2) \Rightarrow y - 5 = 3(x - 2)$

Reflecting these lines about $x + 2y = 1$, we get $y = -3x + 11 \Rightarrow y = \left(\frac{1}{3}\right)x + \frac{13}{3}$

248. Given ray $y = \frac{2x}{3} - 4$. Point of incidence lies on $x$-axis so $y = 0$

$\Rightarrow 0 = \frac{2x}{3} - 4 \Rightarrow x = 6$. Point of incidence is $(6, 0)$

Slope of incident ray $m = \frac{2}{3}$. Reflection from $x$-axis changes slope to $-m$. $\Rightarrow m' = -\frac{2}{3}$

Equation of reflected ray is $y - 0 = \left(-\frac{2}{3}\right)(x - 6) \Rightarrow y = -\frac{2x}{3} + 4$

249. Given point is $M(-2, 3)$ and $\tan\alpha = 3$. Slope of incident ray $m = 3$.

Equation of incident ray $y - 3 = 3(x + 2) \Rightarrow y = 3x + 9$

Point of incidence on $x$-axis so $y = 0, 0 = 3x + 9 \Rightarrow x = -3$

Point of incidence $(-3, 0)$. Reflection from $x$-axis changes slope to $-3$.

Equation of reflected ray is $y - 0 = -3(x + 3) \Rightarrow y = -3x - 9$

250. The point $B(7, 2)$ is reflected across the line $2x + y - 6 = 0$ where $a = 2$, $b = 1$, and $c = -6$.

Compute $ax_0 + by_0 + c = 2 * 7 + 1 * 2 - 6 = 10$ and $a^2 + b^2 = 2^2 + 1^2 = 5$.

The reflected point is found using $x' = x_0 - \frac{2a(ax_0+by_0+c)}{a^2+b^2}$ and $y' = y_0 - \frac{2b(ax_0+by_0+c)}{a^2+b^2}$.

Thus, $x' = 7 - \frac{2*2*10}{5} = -1$ and $y' = 2 - \frac{2*1*10}{5} = -2$, so the reflected point is $B'(-1, -2)$.

The incident beam is the line through $A(3, 10)$ and $B'(-1, -2)$.

The slope is $m = \frac{-2-10}{-1-3} = 3$.

The equation of the incident beam is $y - 10 = 3(x - 3)$, which simplifies to $y = 3x + 1$.

The point of incidence is obtained by solving $y = 3x + 1$ with $2x + y - 6 = 0$.

Substitution gives $2x + (3x + 1) - 6 = 0$, hence $5x - 5 = 0$.

Thus, $x = 1$ and $y = 4$, so the point of incidence is $P(1, 4)$.

The reflected beam passes through $P(1, 4)$ and $B(7, 2)$.

The slope is $m = \frac{2-4}{7-1} = -\frac{1}{3}$.

The equation of the reflected beam is $y - 4 = \left(-\frac{1}{3}\right) * (x - 1)$, which simplifies to $y = \left(-\frac{x}{3}\right) + \frac{13}{3}$.

251. $A = (0, 12)$ and $B = (8, 0)$. Midpoint is $M = (4, 6)$

Slope is $AB = \frac{0-12}{8-0} = -\frac{3}{2}$. Perpendicular's slope is $\frac{2}{3}$

Perpendicular bisector's equation is $y - 6 = \frac{2}{3}(x - 4) \Rightarrow y = \frac{2}{3}x + \frac{10}{3}$

Line through $(0, -1)$ parallel to $x$-axis is $y = -1$

Intersection point is given by $-1 = \frac{2}{3}x + \frac{10}{3} \Rightarrow x = -\frac{13}{2} \Rightarrow C = \left(-\frac{13}{2}, -1\right)$

$\Delta = \frac{1}{2}\left|0(0 - (-1)) + 8((-1) - 12) + \left(-\frac{13}{2}\right)(12 - 0)\right| = 91.$

252. Condition for concurrency is $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

$= a(c^2 - ab) - b(bc - a^2) + c(b^2 - ca) = ac^2 - a^2b - b^2c + a^2b + b^2c - ac^2 = 0.$

253. From the previous problem it is clear that these lines will be concurrent as the coefficients are cyclic in nature.

254. Applying $R_1 \to R_1 + R_2 + R_3$ makes the determinant zero. This is the condition for concurrency of the lines

$(x_2 - x_3)x + (y_2 - y_3)y - [x_1(x_2 - x_3) + y_1(y_2 - y_3)] = 0,$

$(x_3 - x_1)x + (y_3 - y_2)y - [x_2(x_3 - x_1) + y_2(y_3 - y_1)] = 0,$ and

$(x_1 - x_2)x + (y_2 - y_2)y + [x_3(x_1 - x_2) + y_3(y_1 - y_2)] = 0.$

Also from first line $y - y_1 = \frac{x_2 - x_3}{y_2 - y_3}(x - x_1)$, which is the altitude through $(x_1, y_1)$.

255. Given equation is $(2x + 3y - 5)\cos\theta + (3x - 5y + 2)\sin\theta = 0$ which represents the following lines passing through a common point $2x + 3y - 5 = 0$ and $3x - 5y + 2 = 0$.

Solving both the equations we get the fixed point as $(1, 1)$

The line is $x + y = \sqrt{2} \Rightarrow 1 + 1 - \sqrt{2} = 2 - \sqrt{2} \Rightarrow d = \frac{2-\sqrt{2}}{2}$

$x' = \sqrt{2} - 1$ and $y' = \sqrt{2} - 1$

256. The three lines are $y = m_1x + \frac{a}{m_1}$, $y = m_2x + \frac{a}{m_2}$, and $y = m_3x + \frac{a}{m_3}$.

Assume that the orthocenter is $(h, k)$. The altitude to the line $y = m_1x + \frac{a}{m_1}$ has slope $-\frac{1}{m_1}$

$\Rightarrow k - \left(m_2h + \frac{a}{m_2}\right) = -\frac{1}{m_1}(h - h)$. This gives $k = m_2h + \frac{a}{m_2}$

Similarly using other vertices and altitudes we obtain a symmetric system.

Substituting $h = -a$ into each line gives $k = -am_1 + \frac{a}{m_1}$, $k = -am_2 + \frac{a}{m_2}$, and $k = -am_3 + \frac{a}{m_3}$.

Adding the three expressions $3k = -a(m_1 + m_2 + m_3) + a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}\right)$

Using identity $m_1 + m_2 + m_3 = -\frac{1}{m_1m_2m_3}$, $3k = a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1m_2m_3}\right)$

$\Rightarrow k = a\left(\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_1m_2m_3}\right)$

257. Let the points be $P(x_1, y_1)$, $Q(x_2, y_2)$, and $R(x_3, y_3)$.

Given $x_1y_1 = c^2$, $x_2y_2 = c^2$, and $x_3y_3 = c^2$.

$\Rightarrow y_1 = \frac{c^2}{x_1}$, $\Rightarrow y_2 = \frac{c^2}{x_2}$, and $\Rightarrow y_3 = \frac{c^2}{x_3}$

The slope of $QR$ is

$\frac{y_2-y_3}{x_2-x_3} = \frac{\frac{c^2}{x_2}-\frac{c^2}{x_3}}{x_2-x_3} = c^2\frac{x_3-x_2}{x_2x_3(x_2-x_3)} = -\frac{c^2}{x_2x_3}$

The slope of altitude from $P$ is $\frac{x_2x_3}{c^2}$

Equation of altitude from $P$ is $y-\frac{c^2}{x_1} = \left(\frac{x_2x_3}{c^2}\right)(x-x_1)$

Similarly altitude from $Q$ is $y-\frac{c^2}{x_2} = \left(\frac{x_1x_3}{c^2}\right)(x-x_2)$

Solving these gives the orthocenter

$x = -\frac{c^2}{x_1x_2x_3}$ and $y = -x_1x_2x_3$

Multiplying gives the desired result.

258. Given $A = (3,2)$ and $B = (5,1)$. Let $P = (x,y)$.

Since triangle $ABP$ is equilateral $(x-3)^2+(y-2)^2 = 5$ and $(x-5)^2+(y-1)^2 = 5$.

$\Rightarrow x^2+y^2-6x-4y+8 = 0$ and $x^2+y^2-10x-2y+21 = 0$

Subtracting, $-4x+2y+13 = 0 \Rightarrow y = 2x-\frac{13}{2}$

Substituting into first equation $x^2+\left(2x-\frac{13}{2}\right)^2-6x-4\left(2x-\frac{13}{2}\right)+8 = 0$

$4x^2-32x+61 = 0 \Rightarrow x = 4\pm\frac{\sqrt{3}}{2} \Rightarrow y = 2x-\frac{13}{2}$

Points are $P_1 = \left(4+\frac{\sqrt{3}}{2}, \frac{3}{2}+\sqrt{3}\right)$ and $P_2 = \left(4-\frac{\sqrt{3}}{2}, \frac{3}{2}-\sqrt{3}\right)$.

Point away from origin $P = \left(4+\frac{\sqrt{3}}{2}, \frac{3}{2}+\sqrt{3}\right)$.

In an equilateral triangle orthocenter coincides with centroid $H = \left(4+\frac{\sqrt{3}}{2}, \frac{3}{2}+\sqrt{3}\right)$

259. Given $A = (x_1, x_1\tan\alpha_1)$, $B = (x_2, x_2\tan\alpha_2)$, $C = (x_3, x_3\tan\alpha_3)$

Rewriting each point $A = \left(x_1, x_1\sin\frac{\alpha_1}{\cos}\alpha_1\right) = (r_1\cos\alpha_1, r_1\sin\alpha_1)$, where $r_1 = x_1\sec\alpha_1$

Similarly $B = (r_2\cos\alpha_2, r_2\sin\alpha_2)$ and $C = (r_3\cos\alpha_3, r_3\sin\alpha_3)$

Since circumcenter is origin, all vertices lie on a circle centered at origin

So $r_1 = r_2 = r_3 = r$

Thus, $A = (r\cos\alpha_1, r\sin\alpha_1)$, $B = (r\cos\alpha_2, r\sin\alpha_2)$, and $C = (r\cos\alpha_3, r\sin\alpha_3)$

Orthocenter $H(\overline{x},\overline{y})$ satisfies $H = A+B+C$

So $\overline{x} = r(\cos\alpha_1+\cos\alpha_2+\cos\alpha_3)$, $\overline{y} = r(\sin\alpha_1+\sin\alpha_2+\sin\alpha_3)$

$\overline{y}(\cos\alpha_1+\cos\alpha_2+\cos\alpha_3) = r(\sin\alpha_1+\sin\alpha_2+\sin\alpha_3)(\cos\alpha_1+\cos\alpha_2+\cos\alpha_3)$

Similarly $\overline{x}(\sin\alpha_1+\sin\alpha_2+\sin\alpha_3) = r(\cos\alpha_1+\cos\alpha_2+\cos\alpha_3)(\sin\alpha_1+\sin\alpha_2+\sin\alpha_3)$

Both are equal. Hence, $\overline{y}(\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3) = \overline{x}(\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3)$.

260. We find the point of intersection of first two lines $x + ly = l^2$ and $x + my = m^2$

Subtracting yields $(l - m)y = l^2 - m^2 = (l - m)(l + m) \Rightarrow y = l + m$

$\Rightarrow x = l^2 - l(l + m) = -lm$

So $A = (-lm, l + m)$. Similarly $B = (-mn, m + n)$ and $C = (-nl, n + l)$.

$\Delta = \frac{1}{2} \mid x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \mid$

$= \frac{1}{2} \mid (-lm)((m + n) - (n + l)) + (-mn)((n + l) - (l + m)) + (-nl)((l + m) - (m + n)) \mid$

$= \frac{1}{2} \mid (l - m)(m - n)(n - l) \mid$

Slope of $BC$ is $m_{BC} = \frac{(n+l)-(m+n)}{-nl+mn} = \frac{l-m}{n(m-l)} = -\frac{1}{n}$

So altitude from $A$ has slope $n$. $\Rightarrow y - (l + m) = n(x + lm)$

Similarly altitude from $B$ has slope $l$. $\Rightarrow y - (m + n) = l(x + mn)$

From first $y = nx + nlm + l + m$, From second $y = lx + lmn + m + n$

$\Rightarrow nx + nlm + l + m = lx + lmn + m + n \Rightarrow (n - l)x = n - l \Rightarrow x = 1$

$\Rightarrow y = n + nlm + l + m = l + m + n + lmn$.

261. Slope of line $a_1x + b_1y = 1$ is $-\frac{a_1}{b_1}$. Slope of line $a_2x + b_2y = 1$ is $-\frac{a_2}{b_2}$. Slope of line $a_3x + b_3y = 1$ is $-\frac{a_3}{b_3}$

Vertex opposite first side is intersection of $a_2x + b_2y = 1$ and $a_3x + b_3y = 1$.

Call this point $A$. Altitude from $A$ passes through origin. So line joining $A$ and $(0, 0)$ is perpendicular to side $a_1x + b_1y = 1$

Slope of line $OA$ equals slope of line through origin and $A$.
Using property of perpendicular lines $m_{OA} * m_1 = -1$. So $m_{OA} = \frac{b_1}{a_1}$

Now find slope of $OA$. Point $A$ satisfies both equations
$a_2x + b_2y = 1$ and $a_3x + b_3y = 1$

$\Rightarrow a_2b_3x + b_2b_3y = b_3$ and $a_3b_2x + b_3b_2y = b_2$

$\Rightarrow x = \frac{b_3-b_2}{a_2b_3-a_3b_2}$

Similarly $a_2a_3x + b_2a_3y = a_3$ and $a_3a_2x + b_3a_2y = a_2$

$\Rightarrow y = \frac{a_3-a_2}{b_2a_3-b_3a_2}$

Slope of $OA$ $m_{OA} = \frac{y}{x} = \frac{a_3-a_2}{b_3-b_2}$

Since $OA$ is perpendicular to first side, therefore $m_{OA} * \left(-\frac{a_1}{b_1}\right) = -1$

$\Rightarrow \frac{a_3-a_2}{b_3-b_2} * \left(-\frac{a_1}{b_1}\right) = -1$

Simplifying $a_1(a_3 - a_2) = b_1(b_3 - b_2)$

$\Rightarrow a_1a_2 + b_1b_2 = a_1a_3 + b_1b_3$

Similarly by symmetry $a_2a_3 + b_2b_3 = a_2a_1 + b_2b_1$.

262. Let $A, B, C$ be vertices of a triangle. Let $D, E, F$ be midpoints of $BC, CA, AB$ respectively.

So $D = \left(\frac{x_B+x_C}{2}, \frac{y_B+y_C}{2}\right)$, $E = \left(\frac{x_C+x_A}{2}, \frac{y_C+y_A}{2}\right)$ and $F = \left(\frac{x_A+x_B}{2}, \frac{y_A+y_B}{2}\right)$

Centroid of triangle $ABC$ $G = \left(\frac{x_A+x_B+x_C}{3}, \frac{y_A+y_B+y_C}{3}\right)$

Centroid of triangle $DEF$

$x_D + x_E + x_F = \frac{x_B+x_C}{2} + \frac{x_C+x_A}{2} + \frac{x_A+x_B}{2} = \frac{2x_A+2x_B+2x_C}{2} = x_A + x_B + x_C$

Similarly $y_D + y_E + y_F = y_A + y_B + y_C$

So centroid of $DEF$ is $\left(\frac{x_A+x_B+x_C}{3}, \frac{y_A+y_B+y_C}{3}\right)$. Hence. both centroids are same.

Let circumcenter of $ABC$ be $O$. Then, $OA = OB = OC$. So $O$ is equidistant from $A, B, C$.

Thus, $O$ is center of circle passing through $A, B, C$

Since $D, E, F$ are midpoints, triangle $DEF$ is medial triangle.

Each side of $DEF$ is parallel to corresponding side of $ABC$.

So $DE \parallel AB$, $EF \parallel BC$, $FD \parallel CA$.

In medial triangle, each altitude is perpendicular to a side of $DEF$.

But since sides are parallel to $ABC$, these altitudes pass through midpoints and are perpendicular bisectors of $ABC$.

Thus altitudes of $DEF$ pass through $O$. So $O$ lies on all three altitudes of $DEF$.

Hence $O$ is orthocenter of $DEF$.

263. Let $A = (a, \tan\alpha)$, $B = (b, \tan\beta)$, $C = (c, \tan\gamma)$

Since the circumcenter is the origin, we write the points in polar form about the origin.

So we take $a = r\cos\alpha$, $\tan\alpha = r\sin\alpha$, $b = r\cos\beta$, $\tan\beta = r\sin\beta$, and $c = r\cos\gamma$, $\tan\gamma = r\sin\gamma$

Hence, $A = (r\cos\alpha, r\sin\alpha)$, $B = (r\cos\beta, r\sin\beta)$, and $C = (r\cos\gamma, r\sin\gamma)$

Now orthocenter $H(x, y)$ of a triangle with circumcenter at origin satisfies $x = A_x + B_x + C_x$, and $y = A_y + B_y + C_y$

So $x = r(\cos\alpha + \cos\beta + \cos\gamma)$, and $y = r(\sin\alpha + \sin\beta + \sin\gamma)$

Since $\alpha + \beta + \gamma = \pi$, we use identities $\cos\alpha + \cos\beta + \cos\gamma = 1 + 4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)$, and $\sin\alpha + \sin\beta + \sin\gamma = 4\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$

Thus, $x = r\left(1 + 4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\right)$, and $y = 4r\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$

Now considering the expression in the required line

$4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)x - 4y\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$

Substituting $x$ and $y$:

First term is $4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right) * r\left(1 + 4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)\right)$

Second term is $4 * 4r\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right) * \sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)$

So the expression becomes
$4r\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right) +$
$16r\cos^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\gamma}{2}\right) - 16r\sin^2\left(\frac{\alpha}{2}\right)\sin^2\left(\frac{\beta}{2}\right)\sin^2\left(\frac{\gamma}{2}\right)$

Simplification under $\alpha + \beta + \gamma = \pi$, yields $y$

Hence the orthocenter lies on the line $4\left(\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\right)x - 4y\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = y$

264. Let $A$ be intersection of $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$

Let $B$ be intersection of $a_3x + b_3y + c_3 = 0$ and $a_1x + b_1y + c_1 = 0$

Let $C$ be intersection of $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

Let orthocenter be $H(x, y)$

Then $AH$ is perpendicular to $a_1x + b_1y + c_1 = 0$

So slope condition of perpendicular lines gives that the line through $A$ and $H$ satisfies a linear relation obtained by replacing coefficients $(a_1, b_1)$ with $(b_1, -a_1)$ in the direction condition.

Similarly, $BH$ is perpendicular to $a_2x + b_2y + c_2 = 0$ and $CH$ is perpendicular to $a_3x + b_3y + c_3 = 0$.

Solving the system of three altitude equations leads to a linear relation between $a_1x + b_1y + c_1$, $a_2x + b_2y + c_2$, $a_3x + b_3y + c_3$

which is symmetric and reduces to $(a_1x + b_1y + c_1)(a_1a_3 + b_1b_3) = (a_2x + b_2y + c_2)(a_2a_3 + b_2b_3)$

Hence the point $H(x, y)$ satisfies the given equation, so the line passes through the orthocenter of the triangle.

265. Let $A \equiv (1, 1)$ and $B \equiv (2, -1)$.

For $A$ we have $3x + 4y - 6 = -4 < 0$ and for $B$ it is $> 0$.

Hence, the points lie on opposite side of the line.

266. Let the two lines be $L_1 : 2x - 3y + 1 = 0$ and $L_2 : 3x - 5y + 2 = 0$

Evaluate the position of each point with respect to $L_1$ and $L_2$.

For $(0, 0)$ $L_1 = 1 > 0$, $L_2 = 2 > 0$ so sign is $(+, +)$

For $(-1, 1)$ $L_1 = -2 - 3 + 1 = -4 < 0$, $L_2 = -3 - 5 + 2 = -6 < 0$ so sign is $(-, -)$

For $(-7,-4)$ $L_1 = -14+12+1 = -1 < 0$, $L_2 = -21+20+2 = 1 > 0$ so sign is $(-,+)$

For $(9,6)$ $L_1 = 18-18+1 = 1 > 0$, $L_2 = 27-30+2 = -1 < 0$ so sign is $(+,-)$

Thus the four points lie in four different compartments: $(+,+)$, $(-,-)$, $(-,+)$, $(+,-)$

Hence the four points are in four different regions formed by the two lines.

267. We test the origin $(0,0)$ in each side equation.

For $L_1$, $0+1 = 1 > 0$. For $L_2$, $3(0)-4(0)-5 = -5 < 0$. For $L_3$, $5(0)+12(0)-27 = -27 < 0$

So the origin gives signs $(+,-,-)$ with respect to $(L_1, L_2, L_3)$.

$A = L_2 \cap L_3$ $9x - 12y = 15$ Add with $L_3$ $14x = 42 \Rightarrow x = 3$

Substituting $15 - 4y = 5 \Rightarrow y = 2$. So $A = (3,2)$.

$B = L_3 \cap L_1 \Rightarrow x = -1 \Rightarrow 5(-1) + 12y = 27 \Rightarrow y = \frac{8}{3}$. So $B = \left(-1, \frac{8}{3}\right)$.

$C = L_1 \cap L_2 \Rightarrow x = -1 \Rightarrow 3(-1) - 4y = 5 \Rightarrow y = -2$. So $C = (-1,-2)$

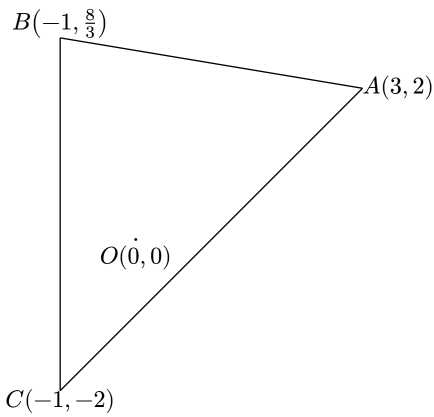
Evaluating sign of origin w.r.t. each side

$L_1(0,0) > 0$, $L_2(0,0) < 0$, $L_3(0,0) < 0$

So origin lies in the region determined by $(+,-,-)$.

Since all three vertices lie on consistent opposite half-planes and the origin satisfies one side positive and two negative, the origin lies inside the triangle region formed by these lines.

Hence, the origin lies inside the triangle.




268. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Let the line $ax + by + c = 0$ cut $AB$ at $P$ in the ratio $m : n$.

So by section formula $P = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$

Since $P$ lies on the line $a\left(\frac{mx_2+nx_1}{m+n}\right)+b\left(\frac{my_2+ny_1}{m+n}\right)+c=0$

Multiplying by $(m+n)$ $m(ax_2+by_2)+n(ax_1+by_1)+c(m+n)=0$

Grouping terms $m(ax_2+by_2+c)+n(ax_1+by_1+c)=0$

Hence $\frac{m}{n}=-\frac{ax_1+by_1+c}{ax_2+by_2+c}$

The quantities $ax_1+by_1+c$ and $ax_2+by_2+c$ are signed values

If the two points lie on opposite sides of the line, these expressions have opposite signs.

Thus their ratio is negative, and the minus sign ensures the ratio $\frac{m}{n}$ remains positive for internal division.

If both points lie on the same side, the ratio becomes negative, indicating external division.

Hence the minus sign accounts for the signed nature of the expressions and distinguishes internal and external division.

269. Using the directed segment ratio result:

For point $P$ on $BC$, $B\frac{P}{P}C=-\left(\frac{b(B)}{b(C)}\right)$ where $b(X)=ax_X+by_X+c$

$\frac{BP}{PC}=-\left(\frac{f(B)}{f(C)}\right)$

Similarly $\frac{CQ}{QA}=-\left(\frac{f(C)}{f(A)}\right)$ and $\frac{AR}{RB}=-\left(\frac{f(A)}{f(B)}\right)$ where $f(X)=ax+by+c$ evaluated at point $X$.

$\frac{BP}{PC}*\frac{CQ}{QA}*\frac{AR}{RB}=\left(-\frac{f(B)}{f(C)}\right)\left(-\frac{f(C)}{f(A)}\right)\left(-\frac{f(A)}{f(B)}\right)=-1$

So $\frac{BP}{PC}*\frac{CQ}{QA}*\frac{AR}{RB}+1=0\Rightarrow BP.CQ.AR+PC.QA.BR=0.$

**Aliter:** Let $B$ be the origin such that $BC=k$ and $C\equiv(k,0)$. Let $A\equiv(\alpha,\beta)$.

Let $\frac{BP}{PC}=m, \frac{CQ}{QA}=-n$ and $\frac{AR}{RB}=p$, where $m,n,p>0$.

$\Rightarrow P\equiv\left(\frac{mk}{m+1},0\right), Q\equiv\left(\frac{n\alpha-k}{n-1},\frac{n\beta}{n-1}\right)$ and $R\equiv\left(\frac{\alpha}{p+1},\frac{\beta}{p+1}\right)$

Since $P,Q,R$ are collinear(they are on the same line $L$), therefore

$\begin{vmatrix}\frac{mk}{m+1} & 0 & 1\\ \frac{n\alpha-k}{n-1} & \frac{n\beta}{n-1} & 1\\ \frac{\alpha}{p+1} & \frac{\beta}{p+1} & 1\end{vmatrix}=0\Rightarrow mnp=1\Rightarrow m(-n)p=-1$. Hence proved.

270. Let the triangle be $ABC$ equation of whose sides $CA$, $AB$ and $BC$ are respectively $3x+y+2=0, 3y-2x=5$ and $x+4y=14$.

Let $L_1$ cut $x$ and $y$ axes at $L$ and $M$ respectively. Then $M\equiv(0,-2)$ and $L\equiv\left(-\frac{2}{3},0\right)$

Let $L_2$ cut $x$ and $y$ axes at $R$ and $P$ respectively. Then $R\equiv\left(-\frac{5}{2},0\right)$ and $P\equiv\left(0,\frac{5}{3}\right)$

Let $L_3$ cut $x$ and $y$ axes at $S$ and $Q$ respectively. Then $S\equiv(14,0)$ and $Q\equiv\left(0,\frac{7}{2}\right)$.

Clearly, point $(0,\beta)$ lies on the $y$-axis. If this point has to be inside the triangle $ABC$ then $\frac{5}{3}\le\beta\le\frac{7}{2}$.

271. Let $A = (2, 3)$ and $B = (-2, 6)$ be consecutive vertices of a rhombus. Given that two sides are parallel to $2x + y = 1$, so slope is $-2$.

So one pair of opposite sides has slope $-2$ and the other pair has slope $-\frac{3}{4}$ from $AB$.

Slope of $AB$ is $m_{AB} = \frac{6-3}{-2-2} = -\frac{3}{4}$

So side directions are fixed.

Equation of side through $A$ is $y - 3 = -2(x - 2) \Rightarrow y = -2x + 7$

Equation of side through $B$ is $y - 6 = -2(x + 2) \Rightarrow y = -2x + 2$

Distance condition for fourth vertex on line through $A$. Let $D = (x, y)$ lie on $y = -2x + 7$

Rhombus has $AD = AB$. $AB^2 = (-4)^2 + 3^2 = 25 \Rightarrow (x - 2)^2 + (y - 3)^2 = 25$

Substitute $y = -2x + 7 \Rightarrow (x - 2)^2 + (-2x + 7 - 3)^2 = 25 \Rightarrow x^2 - 4x - 1 = 0 \Rightarrow x = 2 \pm \sqrt{5}, \Rightarrow y = 3 \mp 2\sqrt{5}$

**Case 1:** $x = 2 + \sqrt{5}$, $y = 3 - 2\sqrt{5}$

**Case 2:** $x = 2 - \sqrt{5}$, $y = 3 + 2\sqrt{5}$

Only the configuration consistent with convex ordering is $D = \left(2 + \sqrt{5}, 3 - 2\sqrt{5}\right)$

$A + C = B + D \Rightarrow C = B + D - A \Rightarrow C = (-2, 6) + \left(2 + \sqrt{5}, 3 - 2\sqrt{5}\right) - (2, 3)$

$\Rightarrow x = -2 + 2 + \sqrt{5} - 2 = -2 + \sqrt{5}$ and $y = 6 + 3 - 2\sqrt{5} - 3 = 6 - 2\sqrt{5}$.

272. For $(3, -4)$ we have $3 * 3 - 4 * (-4) - 8 = 17 > 0$ and for $(2, 6)$ we have $2 * 3 - 4 * 6 - 8 = -26 < 0$.

Thus, the points are on opposite sides of the given line.

273. For $(2, -1)$ we have $3 * 2 + 4 * -1 - 6 = -4 < 0$ and for $(1, 1)$ we have $3 * 1 + 4 * 1 - 6 = 1 > 0$.

Thus, the points are on opposite sides of the given line.

274. For $(3, 4)$ we have $6 * 3 + 4 - 1 = 21 > 0$, and for $(-1, 1)$ we have $6 * (-1) + 1 - 1 = -6 < 0$.

Thus, the points are on opposite sides of the given line.

275. Intersect $x - y = 2$ with $2x + y = 7$. From $x - y = 2$, we get $y = x - 2$.

Substitute into $2x + y = 7$ yields $x = 3$ and $y = 3 - 2 = 1$. So the first point is $(3, 1)$.

Next, intersect $x - y = 2$ with $2x + y = 16$. Again, $y = x - 2$.

Substitute into $2x + y = 16 \Rightarrow 2x + (x - 2) = 16 \Rightarrow x = 6$. Then $y = 6 - 2 = 4$. So the second point is $(6, 4)$.

For point $(3, 1)$: $x + y = 3 + 1 = 4 < 5$ For point $(6, 4)$: $x + y = 6 + 4 = 10 > 5$

Since one point gives a value less than 5 and the other greater than 5, the two points lie on opposite sides of the line $x + y = 5$.

276. Length of perpendicular is $\frac{|3*4-5*5+7|}{\sqrt{3^2+(-5)^2}} = \frac{6}{\sqrt{34}}$.

277. Given lines are $x + 2y = 5$ and $x - 3y = 7$. We solve the equations to get point of intersection.

$(x + 2y) - (x - 3y) = 5 - 7 \Rightarrow 5y = -2 \Rightarrow y = -\frac{2}{5}$.

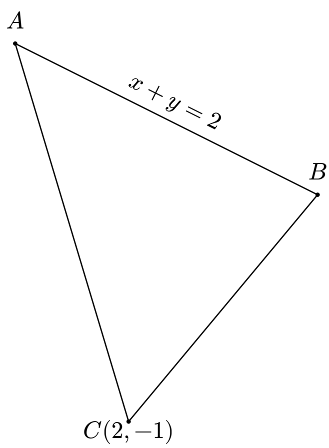
Substitute into $x + 2y = 5$ yields $x + 2\left(-\frac{2}{5}\right) = 5 \Rightarrow x = 5 + \frac{4}{5} = \frac{29}{5}$.

So the intersection point is $\left(\frac{29}{5}, -\frac{2}{5}\right)$.

Equation of the required line with slope 5 is $y - \left(-\frac{2}{5}\right) = 5\left(x - \frac{29}{5}\right) \Rightarrow y + \frac{2}{5} = 5x - 29 \Rightarrow y = 5x - \frac{147}{5} \Rightarrow 25x - 5y - 147 = 0$.

Distance from point $(1, 2)$ to this line is $d = \frac{|25(1)-5(2)-147|}{\sqrt{25^2+(-5)^2}} = \frac{132}{\sqrt{65}}$.

278.




In an equilateral triangle, the perpendicular distance from the vertex to the base is the altitude, and it relates to the side length $a$ by $h = \frac{\sqrt{3}}{2}a$.

First find the distance from the vertex $(2, -1)$ to the line $x + y = 2$.

Writing the line as $x + y - 2 = 0$.

Distance is $d = \frac{|2+(-1)-2|}{\sqrt{1^2+1^2}} = \frac{|-1|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$. So the altitude is $h = \frac{1}{\sqrt{2}}$.

Now use $h = \frac{\sqrt{3}}{2}a$: $\frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}a$

Solving for $a$: $a = \frac{2}{\sqrt{3}} * \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{6}}$.

279. The equation of straight line in the intercept form is $\frac{x}{a} + \frac{y}{b} - 1 = 0$.

The length of the perpendicular drawn is $\frac{|\frac{0}{a}+\frac{0}{b}-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

280. $p = \frac{|0.\sin\theta - 0.\cos\theta - \frac{a}{2}\sin 2\theta|}{\sqrt{\sin^2\theta + \cos^2\theta}} = |\frac{a}{2}\sin 2\theta| \Rightarrow p^2 = \frac{a^2}{4}\sin^2 2\theta$

$p' = \frac{|0.\cos\theta - 0.\sin\theta - a\cos 2\theta|}{\sqrt{\sin^2\theta + \cos^2\theta}} \Rightarrow p^2 = a^2\cos^2 2\theta$

$\Rightarrow 4p + p'^2 = a^2.$

281. Let $b$ be intercept on $y$-axis then intercept on $x$-axis will be $2b$. Thus, equation of the line is $\frac{x}{2b} + \frac{y}{b} - 1 = 0 \Rightarrow x + 2y - 2b = 0$

Length of perpendicular $1 = \frac{|0+0-2b|}{\sqrt{1+4}} \Rightarrow b = \pm\frac{\sqrt{5}}{2}.$

Hence, the equations are $x + 2y \pm \sqrt{5} = 0.$

282. Let $(\alpha, \beta)$ be any point on the first line. Then $a\alpha + b\beta + c = 0.$

Distance of $(\alpha, \beta)$ from the second line is given by $\frac{|a\alpha + b\beta + d|}{\sqrt{a^2+b^2}} = \frac{|-c+d|}{\sqrt{a^2+b^2}}.$

283. Let $p_1, p_2, p_3$ be the length of perpendiculars from the points $(m^2, 2m), (mn, m + n)$ and $(n^2, 2n)$ to the line $x\cos\theta + y\sin\theta + \frac{\sin^2\theta}{\cos\theta}.$

$p_1 = \frac{m^2\cos\theta + 2m\sin\theta + \frac{\sin^2\theta}{\cos\theta}}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{(m\cos\theta + \sin\theta)^2}{\cos\theta}.$

$p_2 = \frac{mn\cos^2\theta + m\sin\theta\cos\theta + n\sin\theta\cos\theta + \sin^2\theta}{\cos\theta} = \frac{(m\cos\theta + \sin\theta)(n\cos\theta + \sin\theta)}{\cos\theta}$

$p_3 = \frac{(n\cos\theta + \sin\theta)^2}{\cos\theta}.$

Clearly $p_1 p_2 = p_3$ i.e. length of perpendiculars are in G.P.

284. Let the tower be at $(0, 0)$. The towns are at $(5, 0)$ and $\left(0, \frac{5}{2}\right)$.

Slope of the road is $m = \frac{\frac{5}{2} - 0}{0 - 5} = -\frac{1}{2}$

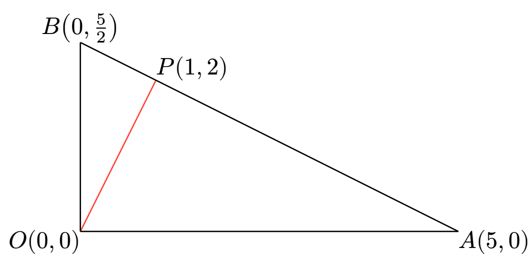
Equation of line is $y = \left(-\frac{1}{2}\right)(x - 5) \Rightarrow x + 2y - 5 = 0$

The nearest point on this line from the origin is the foot of perpendicular.

Using formula for foot from $(0, 0)$ to $ax + by + c = 0$ is

$(x, y) = \left(-a\frac{c}{a^2+b^2}, -b\frac{c}{a^2+b^2}\right)$ Here $a = 1, b = 2, c = -5.$

So, $x = \frac{5}{5} = 1$ and $y = \frac{10}{5} = 2$. Thus, the rest house should be at $(1, 2)$.



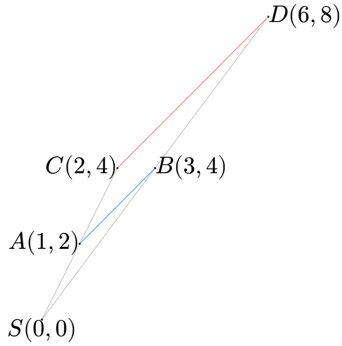

285. Let the line be $ax + by + c = 0$ and the fixed points be $(x_r, y_r; r = 1, 2, 3, \ldots, n)$.

Given $\sum_{r=1}^{n} \frac{ax_r + by_r + c}{\sqrt{a^2 + b^2}} = 0 \Rightarrow \sum_{r=1}^{n}(ax_r + by_r + c = 0)$

$\Rightarrow a(x_1 + x_2 + \cdots + x_n) + b(y_1 + y_2 + \cdots + y_n) + cn = 0$

Thus, the line passes through the fixed poin $\left(\frac{x_1+x_2+\cdots+x_n}{n}, \frac{y_1+y_2+\cdots+y_n}{n}\right)$.

286.




Let $PQ$ be the wall and $CD$ be the shadow of the rod $AB$ on the wall, then $PQ \parallel AB$

Equation of rod $AB$ is $x - y + 1 = 0$, therefore, equation of $PQ$ is $x - y = k$

Length of perpendicular from $S$ to $AB = p_1 = \left|\frac{0-0+1}{\sqrt{1+(-1)^2}}\right| = \frac{1}{\sqrt{2}}$

Length of perpendicular from $S$ to $CD = p_2 = |k\frac{|}{\sqrt{2}}$

From the question $2p_1 = p_2 \Rightarrow k = \pm 2$. If $k = -2$ then $S$ lies on opposite sides of the rod and the wall, which is not possible. Therefore,

$k = 2$ and $CD$ is $x - y + 2 = 0$.

Equation of $SA$ is $y = 2x$ and equation of $SB$ is $y = \frac{4}{3}x$

Solving $CD$ and $SA$ we have $C = (2, 4)$ and solving $CD$ and $SB$ we have $D = (6, 8)$.

$CD = \sqrt{(2-6)^2 + (4-8)^2} = 4\sqrt{2}$.

287. Let $ABCD$ be the parallelogram and $DL \perp AB, BM \perp AD$. Let $\angle DAB = \theta$ then from the right angled $\triangle ADL, DL = AD\sin\theta$

From right-angled $\triangle AMB, BM = AB\sin\theta$

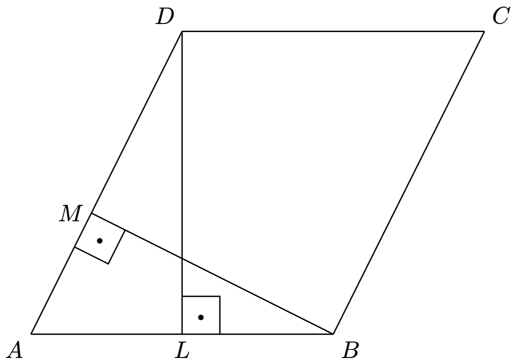
For this it is sufficient to show that it is a rhombus i.e. $AD = AB$.

$\Rightarrow AD\sin\theta = AB\sin\theta \Rightarrow DL = BM$.

Let the given straight lines be $AB, BC, CD, AD$ respectively.

$DL =$ distance between parallel lines $AB$ and $DC = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$ and similarly $BM = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}$.

Thus, $ABCD$ is a rhombus, and hence, the diagonals are perpendicular.




288. The diagram is same as one given in previous problem. Let $ABCD$ be the given parallelogram and the given sides are $AB, BC, CD, AD$ respectively.

$DL = \frac{|a-b|}{\sqrt{1+m^2}}$ and $BM = \frac{|c-d|}{\sqrt{1+m^2}}$

If $\theta$ is the acute angle between $AB$ and $AD$ then $\tan\theta = \frac{|m-n|}{\sqrt{1+mn}}$

$\Rightarrow \sin\theta = \frac{|m-n|}{\sqrt{(m-n)^2+(1+mn)^2}} = \frac{|m-n|}{\sqrt{(1+m^2)(1+n^2)}}$

Area of $\varparallelogram ABCD = AB.DL = AB.AD\sin\theta = \frac{MB.DL}{\sin\theta}$

$= \left|\frac{(a-b)(c-d)}{m-n}\right|$.

289. From first line $x = \frac{21-3y}{2}$. Substitute in second $3\left(\frac{21-3y}{2}\right) - 4y + 11 = 0 \Rightarrow y = 5$. Then $x = \frac{21-15}{2} = 3$

So point of intersection is $(3, 5)$.

Distance is $p = |8x + 6y + 5\frac{|}{\sqrt{8^2+6^2}}$

Substituting $(3, 5)$

$p = |24 + 30 + 5\frac{|}{\sqrt{64+36}} = \frac{59}{10}$.

290. Let the given points be $A(a, b)$ and $B(b, a)$. Slope of $AB$ is $m = \frac{a-b}{b-a} = -1$

So equation is $y - b = -1(x - a) \Rightarrow x + y - (a + b) = 0$

Distance from origin $(0, 0)$ to this line is $d = \frac{|0+0-(a+b)|}{\sqrt{1^2+1^2}} = \frac{|a+b|}{\sqrt{2}}$.

291. Multiply first by 4 to get $8x - 12y = -56$. Multiply second by 3 to get $15x + 12y = 21$

$\Rightarrow x = -\frac{35}{23}$ and $y = \frac{84}{23}$

Line joining origin and $P$ has slope $m = \frac{\frac{84}{23}}{-\frac{35}{23}} = -\frac{12}{5}$. So equation is $y = -\frac{12}{5}x \Rightarrow 12x + 5y = 0$

Distance from $(4, -7)$ to this line $d = |12(4) + 5(-7)\frac{|}{\sqrt{12^2+5^2}} = 1$.

292. Given line is $x + 7y + 2 = 0$. Any line parallel to it is $x + 7y + c = 0$.

Distance from point $(1, -1)$ to this line is

$|1 + 7(-1) + c\frac{|}{\sqrt{1^2+7^2}} = 1 \Rightarrow | - 6 + c| = \sqrt{50}$

So $c - 6 = \pm\sqrt{50}$ and $c = 6 \pm \sqrt{50}$

Hence required lines are $x + 7y + 6 + \sqrt{50} = 0$ and $x + 7y + 6 - \sqrt{50} = 0$.

293. Given line $3x - 4y - 5 = 0$. Let required parallel lines be $3x - 4y + c = 0$

Distance between two parallel lines $|c + 5\frac{|}{\sqrt{3^2+(-4)^2}} = 1 \Rightarrow |c + 5| = 5$

So $c + 5 = \pm 5$ and $c = 0$ or $c = -10$

Hence required lines are $3x - 4y = 0$ and $3x - 4y - 10 = 0$.

294. Let a line through $(0, a)$ be $y - a = m(x - 0)$. So $y = mx + a$.

Rewrite in standard form $mx - y + a = 0$.

Distance from $(2a, 2a)$ is $\frac{|2am-2a+a|}{\sqrt{m^2+1}} = a$

$\Rightarrow (2m - 1)^2 = m^2 + 1 \Rightarrow m = 0$ or $m = \frac{4}{3}$

**Case 1:** $m = 0$ gives $y = a$

**Case 2:** $m = \frac{4}{3}$ gives $y - a = \left(\frac{4}{3}\right)x \Rightarrow 4x - 3y + 3a = 0$.

295. From first$x = 3y - 1$. Substitute in second $2(3y - 1) + 5y - 9 = 0 \Rightarrow 6y - 2 + 5y - 9 = 0 \Rightarrow y = 1 \Rightarrow x = 2$.

So point of intersection is $(2, 1)$. Let required line be$y - 1 = m(x - 2)$. So $mx - y + (1 - 2m) = 0$.

Distance from origin is $\sqrt{5}$So$|1 - 2m\frac{|}{\sqrt{m^2+1}} = \sqrt{5} \Rightarrow m = -2$

Substitute into line equation $y - 1 = -2(x - 2)y = -2x + 5$.

Hence required line is$2x + y - 5 = 0$.

296. From first line $x = y - 1$. Substitute in second line $2(y - 1) - 3y + 5 = 0 \Rightarrow y = 3 \Rightarrow x = 2$

So the point is $(2, 3)$. Let required line be $y - 3 = m(x - 2) \Rightarrow mx - y + (3 - 2m) = 0$

Distance from $(3, 2)$ is $\frac{7}{5}$ $|3m - 2 + 3 - 2m\frac{|}{\sqrt{m^2+1}} = \frac{7}{5} \Rightarrow |m + 1\frac{|}{\sqrt{m^2+1}} = \frac{7}{5}$

$\Rightarrow m = \frac{4}{3} \quad \text{or} \quad m = \frac{3}{4}$

**Case 1:** $y - 3 = \left(\frac{4}{3}\right)(x - 2) \Rightarrow 4x - 3y + 1 = 0$.

**Case 2:** $y - 3 = \left(\frac{3}{4}\right)(x - 2) \Rightarrow 3x - 4y + 6 = 0$.

297. Given that the distance from $(1, 1)$ to the line $ax - by + c = 0$ is 1

So $\frac{|a-b+c|}{\sqrt{a^2+b^2}} = 1$

Squaring both sides $(a-b+c)^2 = a^2+b^2 \Rightarrow a^2+b^2+c^2-2ab+2ac-2bc = a^2+b^2$

$c^2-2ab+2ac-2bc = 0 \Rightarrow c^2+2ac-2bc = 2ab$

Divide throughout by $2abc$, $\frac{c}{2ab}+\frac{1}{b}-\frac{1}{a}=\frac{1}{c}$

Rearranging yields $\frac{1}{c}+\frac{1}{a}-\frac{1}{b}=\frac{c}{2ab}$.

298. Given line is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta = 1$

So $bx\cos\theta+ay\sin\theta-ab=0$

Distance from $(x_1,y_1)$ is $\frac{|bx_1\cos\theta+ay_1\sin\theta-ab|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$

For $\left(\pm\sqrt{a^2-b^2},0\right)$

Product of perpendiculars $= \frac{|\left(b\sqrt{a^2-b^2}\cos\theta-ab\right)\left(-b\sqrt{a^2-b^2}\cos\theta-ab\right)|}{b^2\cos^2\theta+a^2\sin^2\theta}$

$= \frac{|a^2b^2-b^2(a^2-b^2)\cos^2\theta|}{b^2\cos^2\theta+a^2\sin^2\theta} = \frac{b^2(a^2\sin^2\theta+b^2\cos^2\theta)}{b^2\cos^2\theta+a^2\sin^2\theta} = b^2$.

299. Given lines are $4x+3y=11$ and $8x+6y=15$. Rewrite second line $8x+6y=15$ as $4x+3y=\frac{15}{2}$

So the two parallel lines are $4x+3y-11=0$ and $4x+3y-\frac{15}{2}=0$

Distance between parallel lines $= \frac{|(-\frac{15}{2})-(-11)|}{\sqrt{4^2+3^2}} = \frac{7}{10}$

300. Given lines are $2x+3y=19$, $2x+3y+7=0$ and $2x+3y=6$; all have same slope.

Write in standard form $L_1: 2x+3y-19=0$, $L_2: 2x+3y+7=0$ and $L: 2x+3y-6=0$.

Distance between parallel lines $ax+by+c_1=0$ and $ax+by+c_2=0$ is $\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$

Distance of $L_1$ from $L = \frac{|-19-(-6)|}{\sqrt{2^2+3^2}} = \sqrt{13}$

Distance of $L_2$ from $L = \frac{|7-(-6)|}{\sqrt{13}} = \sqrt{13}$

Hence, both distances are equal, so the lines are equidistant from $2x+3y=6$.

301. Distance between parallel lines $= \frac{|c_1-c|}{\sqrt{m^2+1}}$

302. Given sides are $3x-4y=0$ and $4x+3y=0$

These are perpendicular since $3*4+(-4)3=0$. So they are adjacent sides meeting at origin.

Area of square is 25, so side = 5. Distance between each pair of parallel sides is 5.

For line $3x-4y=0$, required parallel side is $3x-4y+c=0$

Distance from origin $\frac{|c|}{\sqrt{3^2+(-4)^2}} = 5 \Rightarrow |c| = 25$

So lines are $3x-4y+25=0$ or $3x-4y-25=0$

For line $4x+3y=0$, required parallel side is $4x+3y+k=0$

Distance from origin $\frac{|k|}{5} = 5 \Rightarrow |k| = 25$

So lines are $4x + 3y + 25 = 0$ or $4x + 3y - 25 = 0$.

Taking consistent pair forming a square, the other two sides are

$3x - 4y + 25 = 0$ and $4x + 3y + 25 = 0$

303. Given lines are $L_1 : ax + by + c = 0$, $L_2 : a_1x + b_1y + c = 0$, $L_3 : ax + by + c_1 = 0$, and $L_4 : a_1x + b_1y + c_1 = 0$

$L_1$ is parallel to $L_3$ and $L_2$ is parallel to $L_4$, so they form a parallelogram.

Length of one pair of opposite sides equals distance between $L_1$ and $L_3 = \frac{|c_1 - c|}{\sqrt{a^2+b^2}}$

Length of the other pair equals distance between $L_2$ and $L_4 = \frac{|c_1 - c|}{\sqrt{a_1^2+b_1^2}}$

Given $a^2 + b^2 = a_1^2 + b_1^2$. So both lengths are equal.

Hence, all sides of parallelogram are equal. Therefore, it is a rhombus.

304. Given lines are $L_1 : 4x + 3y - 6 = 0$ and $L_2 : 5x + 12y + 9 = 0$

Angle bisectors are given by $\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm \frac{5x+12y+9}{\sqrt{5^2+12^2}}$

$\Rightarrow \frac{4x+3y-6}{5} = \pm \frac{5x+12y+9}{13}$

**Case I:** $13(4x + 3y - 6) = 5(5x + 12y + 9) \Rightarrow 9x - 7y - 41 = 0$

**Case II:** $13(4x + 3y - 6) = -5(5x + 12y + 9) \Rightarrow 7x + 9y - 3 = 0$

To identify angles, test origin $(0, 0)$

For $L_1 : -6 < 0$. For $L_2 : 9 > 0$.

So origin lies between opposite signs, hence lies in one of the angles.

Checking bisectors: For $9x - 7y - 41 = 0 \Rightarrow -41 < 0$

For $7x + 9y - 3 = 0 \Rightarrow -3 < 0$

The bisector that preserves sign relation corresponds to angle containing origin.

So angle containing origin is $7x + 9y - 3 = 0$

Hence, acute angle bisector is $7x + 9y - 3 = 0$ and obtuse angle bisector is $9x - 7y - 41 = 0$.

305. Given lines are $L_1 : 3x + 4y - 5 = 0$ and $L_2 : 12x + 5y - 7 = 0$.

The locus of points equidistant from $L_1$ and $L_2$ is given by the angle bisectors: $\frac{3x+4y-5}{5} = \pm \frac{12x+5y-7}{13}$. So we get two bisectors.

**Case I:** $13(3x + 4y - 5) = 5(12x + 5y - 7) \Rightarrow 7x - 9y + 10 = 0$

**Case II:** $13(3x + 4y - 5) = -5(12x + 5y - 7) \Rightarrow 99x + 77y - 100 = 0$

Now observe: The given line $7x - 9y + 10 = 0$ is exactly the angle bisector of $L_1$ and $L_2$.

306. We have already proven this in Equation (1.9).

307. Given triangle sides $L_1 : x+1=0$, $L_2 : 3x-4y-5=0$, and $L_3 : 5x+12y-27=0$.

$A = L_2 \cap L_3$. From $3x-4y=5$ and $5x+12y=27$

$x=3$ and $y=1$. So $A=(3,1)$.

$B = L_3 \cap L_1 \Rightarrow x=-1$ and $y=\frac{8}{3}$. So $B=\left(-1,\frac{8}{3}\right)$.

$C = L_1 \cap L_2 \Rightarrow x=-1$ and $y=-2$. So $C=(-1,-2)$.

$BC=\sqrt{(0)^2+\left(\frac{14}{3}\right)^2}=\frac{14}{3}$, $CA=\sqrt{(4)^2+(3)^2}=5$, and $AB=\sqrt{(4)^2+\left(-\frac{5}{3}\right)^2}=\frac{13}{3}$

Incenter formula is $I=\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}$.

Hence incenter is $\left(\frac{1}{3},\frac{2}{3}\right)$.

308. Let the given opposite sides be $x+y=1$ and $x+y=5$

Distance between the parallel sides is $d=\frac{|5-1|}{\sqrt{1^2+1^2}}=\frac{4}{\sqrt{2}}=2\sqrt{2}$

Let the side length of the rhombus be $a$. Height of rhombus $= a\sin(45\,\text{deg})$

So $a\sin(45^{\circ})=2\sqrt{2}\Rightarrow a=4$

Direction vector of lines $x+y=k$ is $(1,-1)$. Unit vector $=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

Vertex $A$ is $(2,-1)$. Adjacent vertex $B$ is $B=(2,-1)+-4*\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\Rightarrow B=(2,-1)\pm\left(2\sqrt{2},-2\sqrt{2}\right)$

So $B_1=\left(2+2\sqrt{2},-1-2\sqrt{2}\right)$ and $B_2=\left(2-2\sqrt{2},-1+2\sqrt{2}\right)$

Second side makes $45^{\circ}$ with first side. Rotating $(1,-1)$ by $45^{\circ}$ gives direction $(1,0)$.

So adjacent vertex $C$ is $C_1=(6,-1)$ and $C_2=(-2,-1)$

Opposite vertex $D$ is given by $D=B+C-A$

**Case I:** $D=\left(2+2\sqrt{2},-1-2\sqrt{2}\right)+(6,-1)-(2,-1)\Rightarrow D=\left(6+2\sqrt{2},-1-2\sqrt{2}\right)$

**Case II:** $D=\left(2-2\sqrt{2},-1+2\sqrt{2}\right)+(-2,-1)-(2,-1)\Rightarrow D=\left(-2-2\sqrt{2},-1+2\sqrt{2}\right)$

309. Sides are parallel to $y=x+2$ and $y=7x+3$. So slopes are 1 and 7.

Let $A=(0,t)$. Let adjacent vertices be $B=(x_1,y_1)$ on line through $A$ with slope 1 and $D=(x_2,y_2)$ on line through $A$ with slope 7

So $y_1-t=x_1$ and $y_2-t=7x_2$

Thus, $B=(x_1,t+x_1)$ and $D=(x_2,t+7x_2)$

For rhombus, adjacent sides are equal: $AB^2=AD^2\Rightarrow x_1=\pm 5x_2$

Now $C = B + D - A = (x_1 + x_2, t + x_1 + 7x_2)$

Diagonals bisect each other at $(1, 2)$ i.e. Midpoint of $AC$ is $(1, 2)$

$((x_1 + x_2)(2), * t + t + x_1 + 7x_2)(2)) = (1, 2)$

So $x_1 + x_2 = 2 \Rightarrow 2t + x_1 + 7x_2 = 4$

**Case I:** $x_1 = 5x_2 \Rightarrow x_2 = \frac{1}{3}$ and $x_1 = \frac{5}{3}$

Then $2t + \frac{5}{3} + \frac{7}{3} = 4 \Rightarrow t = 0$

So $A = (0, 0)$

**Case II:** $x_1 = -5x_2 \Rightarrow x_2 = -\frac{1}{2}$ and $x_1 = \frac{5}{2}$

Then $2t + \frac{5}{2} - \frac{7}{2} = 4 => t = \frac{5}{2}$.

So $A = \left(0, \frac{5}{2}\right)$

310. We are given the lines $x - 2y + 3 = 0$ and $4x + 2y - 5 = 0$.

The angle bisectors are given by $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$.

For the given lines, $a_1 = 1$, $b_1 = -2$, $c_1 = 3$ and $a_2 = 4$, $b_2 = 2$, $c_2 = -5$.

So, $\frac{x - 2y + 3}{\sqrt{5}} = \pm \frac{4x + 2y - 5}{\sqrt{20}}$.

Since $\sqrt{20} = 2\sqrt{5}$, $\frac{x - 2y + 3}{\sqrt{5}} = \pm \frac{4x + 2y - 5}{2\sqrt{5}}$.

Multiplying both sides by $\sqrt{5}$, $x - 2y + 3 = \pm \frac{4x + 2y - 5}{2}$.

**Case I:** $x - 2y + 3 = \frac{4x + 2y - 5}{2} \Rightarrow 2x + 6y - 11 = 0$.

**Case II:** $x - 2y + 3 = -\frac{4x + 2y - 5}{2} \Rightarrow 6x - 2y + 1 = 0$.

311. We want to prove that the line $6x + 66y - 7 = 0$ bisects the angle between the lines $15x - 18y - 1 = 0$ and $12x + 10y - 3 = 0$.

The equation is $\frac{15x - 18y - 1}{\sqrt{15^2 + (-18)^2}} = \pm \frac{12x + 10y - 3}{\sqrt{12^2 + 10^2}}$

$\sqrt{15^2 + (-18)^2} = \sqrt{225 + 324} = \sqrt{549}$ and $\sqrt{12^2 + 10^2} = \sqrt{144 + 100} = \sqrt{244}$

So, $\frac{15x - 18y - 1}{\sqrt{549}} = \pm \frac{12x + 10y - 3}{\sqrt{244}}$

$\Rightarrow \sqrt{244}(15x - 18y - 1) = \pm \sqrt{549}(12x + 10y - 3)$

$\Rightarrow 30x - 36y - 2 = \pm(36x + 30y - 9)$

Taking plus sign we see that it is the given equation.

312. Rewrite the given lines in standard form: $24x + 7y - 20 = 0, 4x - 3y - 2 = 0$

Let (x, y) be any point on $2x + 11y = 5$.

Distance from first line: $D_1 = \frac{|24x + 7y - 20|}{\sqrt{24^2 + 7^2}} = \frac{|24x + 7y - 20|}{25}$

Distance from second line: $D_2 = \frac{|4x - 3y - 2|}{\sqrt{4^2 + (-3)^2}} = \frac{|4x - 3y - 2|}{5}$

Now use the relation $2x + 11y = 5$.

Then: $24x + 7y - 20 = (24x + 132y - 60) - 125y + 40 = -5(25y - 8)$

Also, $4x - 3y - 2 = (4x + 22y - 10) - 25y + 8 = -(25y - 8)$

Thus, $|24x + 7y - 20| = 5\ |25y - 8|$

$D_1 = 59|25y - 8|\frac{)}{25} = \frac{|25y-8|}{5}$ and $D_2 = \frac{|25y-8|}{5}$

Hence, $D_1 = D_2$.

313. We are given the lines $6x + 8y - 10 = 0$ and $4x - 3y - 7 = 0$.

For a point $(x, y)$ equidistant from the two lines, $\frac{|6x+8y-10|}{\sqrt{6^2+8^2}} = \frac{|4x-3y-7|}{\sqrt{4^2+(-3)^2}}$

$\sqrt{6^2 + 8^2} = \sqrt{36 + 64} = 10$ and $\sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = 5$

So, $\frac{|6x+8y-10|}{10} = \frac{|4x-3y-7|}{5}$

Multiply: $|6x + 8y - 10| = 2 * |4x - 3y - 7| \Rightarrow 6x + 8y - 10 = \pm 2(4x - 3y - 7)$

**Case I:** $6x + 8y - 10 = 8x - 6y - 14 \Rightarrow x - 7y - 2 = 0$

**Case II:** $6x + 8y - 10 = -8x + 6y + 14 \Rightarrow 7x + y - 12 = 0$

Hence, the locus is $x - 7y - 2 = 0$ or $7x + y - 12 = 0$.

314. The given lines are $x + y - 3 = 0$ and $7x - y + 5 = 0$.

For $x + y - 3 = 0$ we have $a_1 = 1$ and $b_1 = 1$ so the value is $\sqrt{2}$.

For $7x - y + 5 = 0$ we have $a_2 = 7$ and $b_2 = -1$ so the value is $\sqrt{50} = 5\sqrt{2}$.

The angle bisectors satisfy $\frac{x+y-3}{\sqrt{2}} = \pm \frac{7x-y+5}{5}\sqrt{2}$

After simplification this becomes $5(x + y - 3) = \pm(7x - y + 5)$

First result $5x + 5y - 15 = 7x - y + 5 \Rightarrow x - 3y + 10 = 0$

Second result $5x + 5y - 15 = -7x + y - 5 \Rightarrow 6x + 2y - 5 = 0$

315. $\frac{3x+4y-11}{\sqrt{3^2+4^2}} = \pm \frac{12x-5y-2}{\sqrt{12^2+(-5)^2}}$.

This simplifies to $\frac{3x+4y-11}{5} = \pm \frac{12x-5y-2}{13}$.

Taking the negative sign gives $13(3x + 4y - 11) = -5(12x - 5y - 2) \Rightarrow 11x + 3y - 17 = 0$.

The slopes of the given lines are $-\frac{3}{4}$ and $\frac{12}{5}$, which form an obtuse angle. Therefore, the bisector corresponding to the negative sign represents the acute angle.

Hence, the bisector of the acute angle is $11x + 3y = 17$.

316. $\frac{x-2y+4}{\sqrt{5}} = \frac{\pm(4x-3y+2)}{5}$.

Taking the positive case gives $5(x - 2y + 4) = \sqrt{5}(4x - 3y + 2)$.

Taking the negative case givesv $5(x - 2y + 4) = -\sqrt{5}(4x - 3y + 2)$.

The slopes of the given lines are $\frac{1}{2}$ and $\frac{4}{3}$, which form an acute angle. Therefore, the obtuse angle is the supplementary angle, and its bisector corresponds to the negative case.

Hence, the equation of the bisector of the obtuse angle is

$5(x-2y+4)=-\sqrt{5}(4x-3y+2)$.

317. The given lines are $x+y-2=0$ and $x-y-3=0$.

These two lines divide the plane into four compartments depending on the signs of $x+y-2$ and $x-y-3$.

Evaluating both expressions at each point.

For $(1,0)$: $x+y-2=1+0-2=-1<0$ and $x-y-3=1-0-3=-2<0$.

For $(2,3)$: $x+y-2=2+3-2=3>0$ and $x-y-3=2-3-3=-4<0$.

For $(1,-4)$: $x+y-2=1-4-2=-5<0$ and $x-y-3=1-(-4)-3=2>0$.

For $(8,1)$: $x+y-2=8+1-2=7>0$ and $x-y-3=8-1-3=4>0$.

Each point gives a distinct sign combination for $(x+y-2, x-y-3)$.

Thus, the four points lie in four different compartments formed by the given lines.

318. For $7x-5y-11=0$: $7(0)-5(0)-11=-11<0$.

For $8x+3y+31=0$: $8(0)+3(0)+31=31>0$.

For $x+8y-19=0$: $0+8(0)-19=-19<0$.

Thus, the origin lies on the negative side of the first line, the positive side of the second line, and the negative side of the third line.

Now consider a point clearly inside the triangle. Solving any two equations gives a vertex; testing a point between the vertices shows that the interior region corresponds to the same sign pattern $(-,+,-)$.

Since the origin produces this same pattern, it lies inside the triangle.

Hence, the origin lies inside the triangle.

319. These lines are parallel and represent two opposite sides of the square.

The length of the side of the square is equal to the perpendicular distance between the two lines.

Thus, the distance between the lines is

$\frac{|-65-26|}{\sqrt{5^2+(-12)^2}}=\frac{91}{\sqrt{25+144}}=\frac{91}{13}=7.$

So, the side of the square is 7. Hence, the area of the square is $7^2=49$.

320. The given sides of the square are $3x+4y-5=0$ and $3x+4y-15=0$.

These two lines are parallel, so they represent opposite sides of the square.

So the side length is

$\frac{|-5-(-15)|}{\sqrt{3^2+4^2}}=\frac{10}{\sqrt{9+16}}=2.$

Since the third side passes through $(6,5)$ and is perpendicular to the given sides, its equation has the form $4x - 3y + c = 0$.

Substituting $(6,5)$ gives $4(6) - 3(5) + c = 0 \Rightarrow c = -9$.

So one side of the square is $4x - 3y - 9 = 0$.

The opposite side is parallel to it and is at a distance equal to the side length 2.
So the parallel line is $4x - 3y + k = 0$.

Using distance formula between $4x - 3y - 9 = 0$ and $4x - 3y + k = 0$,
$\frac{|k+9|}{\sqrt{4^2+(-3)^2}} = 2$. Thus $k = 1$ or $k = -19$.

Since $(6,5)$ lies on one side, the opposite side is $4x - 3y + 1 = 0$.

Hence, the equations of the remaining two sides are
$4x - 3y - 9 = 0$ and $4x - 3y + 1 = 0$.

321. The given side of the rectangle is $3x - 4y - 10 = 0$, and two vertices are $(2,1)$ and $(2,4)$.

First, observe that the line joining $(2,1)$ and $(2,4)$ is $x = 2$, since both points have the same x-coordinate.
So one side of the rectangle lies on $x = 2$.
Now find the distance between the parallel sides $x = 2$ and the given line $3x - 4y - 10 = 0$.

Taking any point on $x = 2$, for example $(2,1)$, the distance is
$\frac{|3(2)-4(1)-10|}{\sqrt{3^2+(-4)^2}} = \frac{8}{5}$.

So the height of the rectangle is $\frac{8}{5}$.

The length of the rectangle is the distance between $(2,1)$ and $(2,4)$: $\sqrt{(2-2)^2 + (4-1)^2} = 3$.

Hence, the area of the rectangle is $3 \times \frac{8}{5} = \frac{24}{5}$.

One diagonal joins $(2,4)$ to the opposite vertex. The opposite side to $x = 2$ is parallel to it, so the opposite vertical line is $x = 2 + \frac{8}{5} = \frac{18}{5}$ or $x = 2 - \frac{8}{5} = \frac{2}{5}$.

Since $(2,1)$ lies on one side, the opposite vertex to $(2,4)$ is $\left(\frac{18}{5}, 1\right)$.

Now find the equation of the diagonal through $(2,4)$ and $\left(\frac{18}{5}, 1\right)$.

Slope is $\frac{1-4}{\frac{18}{5}-2} = -\frac{3}{\frac{8}{5}} = -\frac{15}{8}$.

Equation is $y - 4 = -\frac{15}{8}(x - 2) \Rightarrow 15x + 8y - 62 = 0$.

Hence, the area of the rectangle is $\frac{24}{5}$ and the required diagonal is $15x + 8y - 62 = 0$.

322. The four lines are $ax + by + c = 0$, $ax + by - c = 0$, $ax - by + c = 0$, and $ax - by - c = 0$.

Opposite sides are parallel since each pair differs only in the constant term. Hence the figure formed is a parallelogram.

Intersect $ax + by + c = 0$ and $ax - by + c = 0$.

We get $ax + by = -c$ and $ax - by = -c$.

Adding gives $2ax = -2c$, so $x = -\frac{c}{a}$. Substituting gives $y = 0$. Thus one vertex is $\left(-\frac{c}{a}, 0\right)$.

Similarly, other vertices are $\left(\frac{c}{a}, 0\right)$, $\left(0, \frac{c}{b}\right)$, and $\left(0, -\frac{c}{b}\right)$.

The diagonals are the lines joining opposite vertices, so they lie along the coordinate axes.

Hence the diagonals are perpendicular, so the parallelogram is a rhombus.

Diagonal 1 has length $|\frac{c}{a} - \left(-\frac{c}{a}\right)| = \frac{2|c|}{|a|}$.

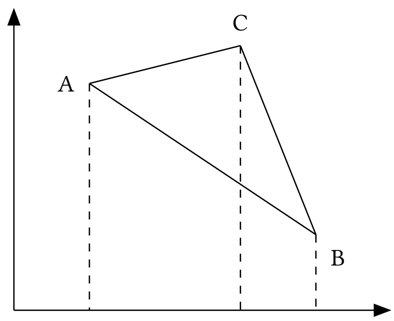
Diagonal 2 has length $|\frac{c}{b} - \left(-\frac{c}{b}\right)| = \frac{2|c|}{|b|}$.

Area of a rhombus is $\left(\frac{1}{2}\right) \times$ (product of diagonals).

So area is $\frac{1}{2} \times \frac{2|c|}{|a|} \times \frac{2|c|}{|b|}$.

This simplifies to $\frac{2c^2}{|ab|}$.

323.




We choose a coordinate system such that the given straight line is the $x$-axis. Then $p, q, r$ become the signed $y$-coordinates of $A, B, C$. So we write $A(x_1, p)$, $B(x_2, q)$, and $C(x_3, r)$.

Now we use the formula for the square of the area of a triangle in coordinate form: $4\Delta^2 = (x_1(q-r) + x_2(r-p) + x_3(p-q))^2 + (p(q-r) + q(r-p) + r(p-q))^2$.

The second bracket simplifies as follows: $p(q-r) + q(r-p) + r(p-q) = 0$.

So we get $4\Delta^2 = (x_1(q-r) + x_2(r-p) + x_3(p-q))^2$.

Now consider the squared side lengths using distance formula:

$a^2 = (x_2 - x_3)^2 + (q-r)^2, \quad b^2 = (x_3 - x_1)^2 + (r-p)^2, \quad c^2 = (x_1 - x_2)^2 + (p-q)^2$.

Now we expand the required expression:

$a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)$.

Substitute $a^2, b^2, c^2$: $\left[(x_2-x_3)^2+(q-r)^2\right](p-q)(p-r)+\left[(x_3-x_1)^2+(r-p)^2\right](q-r)(q-p)+\left[(x_1-x_2)^2+(p-q)^2\right](r-p)(r-q)$.

We separate terms into two groups: those involving $x_i$ and those involving only $p, q, r$.

The pure $p, q, r$ part simplifies to zero due to the identity $(p-q)(p-r)+(q-r)(q-p)+(r-p)(r-q)=0$.

So the expression reduces to $(x_2-x_3)^2(p-q)(p-r)+(x_3-x_1)^2(q-r)(q-p)+(x_1-x_2)^2(r-p)(r-q)$.

Now expand and regroup in terms of $x_1, x_2, x_3$: This becomes $(x_1(q-r)+x_2(r-p)+x_3(p-q))^2$.

But earlier we showed that $(x_1(q-r)+x_2(r-p)+x_3(p-q))^2=4\Delta^2$.

Hence, $a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)=4\Delta^2$.

324. Let the required line pass through $(4,-5)$ with slope $m$. Then its equation is $y+5=m(x-4)$, which gives $mx-y-4m-5=0$.

The distance of the point $(-2,3)$ from this line is $\frac{|m(-2)-3-4m-5|}{\sqrt{m^2+1}}=\frac{|6m+8|}{\sqrt{m^2+1}}$.

Given that this distance equals 12, we write $|6m+8\frac{|}{\sqrt{m^2+1}}=12$.

Squaring both sides gives $(6m+8)^2=144(m^2+1) \Rightarrow 27m^2-24m+20=0$.

The discriminant is $D=(-24)^2-4\times 27\times 20=576-2160=-1584$.

Since $D<0$, there is no real solution for $m$. Hence, no line passing through $(4,-5)$ can have distance 12 from $(-2,3)$.

325. Slope of $BC$ is $\frac{-3-(-1)}{-1-(-3)}=\frac{-2}{2}=-1$.

So equation of $BC$ is $y+1=-1(x+3) \Rightarrow x+y+4=0$.

A line parallel to $BC$ has the form $x+y+k=0$.

The perpendicular distance from origin $(0,0)$ to this line is $\frac{|k|}{\sqrt{1^2+1^2}}=\frac{|k|}{\sqrt{2}}$.

Given this distance is $\frac{1}{2}$, we write $\frac{|k|}{\sqrt{2}}=\frac{1}{2}$.

So $|k|=\frac{\sqrt{2}}{2}$, hence $k=\pm\frac{\sqrt{2}}{2}$.

Thus the required line is $x+y+\frac{\sqrt{2}}{2}=0$ or $x+y-\frac{\sqrt{2}}{2}=0$.

Now check which line intersects segments $OB$ and $OC$.

Line $OB$: slope is $\frac{-1-0}{-3-0}=\frac{1}{3}$, so equation is $y=\left(\frac{1}{3}\right)x$.

Line $OC$: slope is $\frac{-3-0}{-1-0}=3$, so equation is $y=3x$.

On $OB$, substitute $y=\frac{x}{3}$: $x+\frac{x}{3}+\frac{\sqrt{2}}{2}=0$ gives a negative intersection point for $x$, so it lies on segment $OB$.

On $OC$, substitute $y = 3x$: $x + 3x + \frac{\sqrt{2}}{2} = 0$ gives another valid intersection point on segment $OC$.

Thus the required line is $x + y + \frac{\sqrt{2}}{2} = 0$.

326. The center of the square is $C(1, -1)$ and one side is $x - 2y + 12 = 0$.

Distance from $C$ to this line is $\frac{|1-2(-1)+12|}{\sqrt{1^2+(-2)^2}} = \frac{15}{\sqrt{5}} = 3\sqrt{5}$.

So the side length of the square is $2 \times 3\sqrt{5} = 6\sqrt{5}$.

The opposite side is parallel to the given line, so it is $x - 2y + k = 0$.

Distance between parallel sides equals $6\sqrt{5}$, so $\frac{|k-12|}{\sqrt{5}} = 6\sqrt{5}$, giving $|k - 12| = 30$.

Thus $k = 42$ or $k = -18$, so opposite sides are $x - 2y + 42 = 0$ and $x - 2y - 18 = 0$.

Now the other two sides are perpendicular to these, so have form $2x + y + c = 0$.

Distance from center gives $\frac{|2(1)-1+c|}{\sqrt{5}} = 3\sqrt{5}$, so $|1 + c| = 15$.

Thus $c = 14$ or $c = -16$. Hence the remaining sides are $2x + y + 14 = 0$ and $2x + y - 16 = 0$.

327. The given sides are $3x - 2y + 12 = 0$ and $x - 3y + 11 = 0$, and the diagonals intersect at $(2, 2)$, which is the center of the parallelogram.

Opposite sides are parallel to the given ones.

For $3x - 2y + k = 0$, using distance from $(2, 2)$: $|3(2) - 2(2) + k\frac{|}{\sqrt{13}} = \frac{|2+k|}{\sqrt{13}}$.

This equals the distance to $3x - 2y + 12 = 0$, so $(|2 + k|) = 14$, giving $k = 12$ or $k = -16$.

Hence other side is $3x - 2y - 16 = 0$.

For $x - 3y + k = 0$: $\frac{|2-6+k|}{\sqrt{10}} = \frac{|k-4|}{\sqrt{10}}$.

This equals the distance to $x - 3y + 11 = 0$, so $|k - 4| = 7$, giving $k = 11$ or $k = -3$.

Hence other side is $x - 3y - 3 = 0$.

Intersect $3x - 2y + 12 = 0$ and $x - 3y - 3 = 0$ gives $(-6, -3)$. Diagonal through $(2, 2)$ and $(-6, -3)$ is $5x - 8y - 6 = 0$.

Intersect $3x - 2y - 16 = 0$ and $x - 3y + 11 = 0$ gives $(10, 7)$. Diagonal through $(2, 2)$ and $(10, 7)$ is $5x - 8y + 6 = 0$.

Hence other sides are $3x - 2y - 16 = 0$, $x - 3y - 3 = 0$, and diagonals are $5x - 8y - 6 = 0$, $5x - 8y + 6 = 0$.

328. The given parallel lines are $3x + 4y + 2 = 0$, $3x + 4y + 5 = 0$, and $3x + 4y - 5 = 0$.

Since all have the same $3x + 4y$ part, their relative positions depend only on constants $2, 5$, and $-5$.

Clearly $-5 < 2 < 5$, so the line $3x + 4y + 2 = 0$ lies between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$.

Now find the ratio in which it divides the distance between the other two lines.

So total distance between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ is proportional to $|5 - (-5)| = 10$.

Distance from $3x + 4y + 2 = 0$ to $3x + 4y + 5 = 0$ is proportional to $|5 - 2| = 3$.

Distance from $3x + 4y + 2 = 0$ to $3x + 4y - 5 = 0$ is proportional to $|2 - (-5)| = 7$.

Hence the required ratio is $3 : 7$.

Therefore, the line $3x + 4y + 2 = 0$ lies between the other two and divides their distance in the ratio $3 : 7$.

329. The given lines are $x + 2y + 3 = 0$, $x + 2y - 7 = 0$, and $2x - y - 4 = 0$.

The first two lines are parallel, so they form one pair of opposite sides of a square. Their distance is the side length: $|3 - (-7)\frac{|}{\sqrt{1^2+2^2}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$.

Now the other pair of sides is parallel to $2x - y - 4 = 0$, so write $2x - y + k = 0$. Distance between parallel sides is $2\sqrt{5}$: $\frac{|k+4|}{\sqrt{2^2+(-1)^2}} = 2\sqrt{5}$, so $\frac{|k+4|}{\sqrt{5}} = 2\sqrt{5}$, hence $|k + 4| = 10$.

Thus $k = 6$ or $k = -14$.

Therefore, the fourth sides are $2x - y + 6 = 0$ or $2x - y - 14 = 0$.

330. The sides of the triangle are $3x + 4y - 6 = 0$, $12x - 5y - 3 = 0$, and $4x - 3y + 12 = 0$.

Between $3x + 4y - 6 = 0$ and $12x - 5y - 3 = 0$: $\frac{3x+4y-6}{5} = \frac{12x-5y-3}{13}$, giving $13(3x + 4y - 6) = 5(12x - 5y - 3)$, which simplifies to $9x - 77y + 63 = 0$.

Between $12x - 5y - 3 = 0$ and $4x - 3y + 12 = 0$: $\frac{12x-5y-3}{13} = \frac{4x-3y+12}{5}$, giving $5(12x - 5y - 3) = 13(4x - 3y + 12)$, which simplifies to $8x + 19y - 171 = 0$.

Between $4x - 3y + 12 = 0$ and $3x + 4y - 6 = 0$: $\frac{4x-3y+12}{5} = \frac{3x+4y-6}{5}$, giving $4x - 3y + 12 = 3x + 4y - 6$, which simplifies to $x - 7y + 18 = 0$.

Hence the internal bisectors of the triangle are $9x - 77y + 63 = 0$, $8x + 19y - 171 = 0$, and $x - 7y + 18 = 0$.

331. From $3x + 4y = 12$ and $5x + 12y = 20$, we get $A = (4, 0)$.

From $3x + 4y = 12$ and $-7x + 24y = 22$, we get $B = \left(2, \frac{3}{2}\right)$.

From $5x + 12y = 20$ and $-7x + 24y = 22$, we get $C = \left(\frac{18}{17}, \frac{125}{102}\right)$.

Now the side lengths are computed:
$AB = \frac{5}{2}$, $BC = \frac{50}{51}$, $CA = \frac{325}{102}$.

The incenter is given by $I = \frac{aA+bB+cC}{a+b+c}$ where $a = BC$, $b = CA$, $c = AB$.

So $a = \frac{50}{51}$, $b = \frac{325}{102}$, $c = \frac{5}{2}$ and $a + b + c = \frac{340}{51}$.

$x = \frac{a*4+b*2+c*\frac{18}{17}}{\frac{340}{51}}$. This simplifies to $x = \frac{33}{17}$.

$y = \frac{a*0+b*\frac{3}{2}+c*\frac{125}{102}}{\frac{340}{51}}$. This simplifies to $y = \frac{20}{17}$.

332. Let a point $(x, y)$ reflect to $(X, Y)$ in the line $x + y + 1 = 0$.

For reflection in $ax + by + c = 0$, the formula gives

$X = x - 2\frac{a(ax+by+c)}{a^2+b^2}$ and $Y = y - 2\frac{b(ax+by+c)}{a^2+b^2}$.

Here $a = 1$, $b = 1$, $c = 1$, so $a^2 + b^2 = 2$.

Thus $X = x - (x + y + 1) = -y - 1$ and $Y = y - (x + y + 1) = -x - 1$

So the transformation is $x = -Y - 1$, $y = -X - 1$.

Now the given line is $px + qy + r = 0$.

Substitute: $p(-Y - 1) + q(-X - 1) + r = 0 \Rightarrow qX + pY + (p + q - r) = 0$

Hence the reflection of the line is $qx + py + (p + q - r) = 0$.

333. The roads are $x - 2y - 4 = 0$ and $2x - y - 4 = 0$.

Their intersection point is found from $x - 2y = 4$ and $2x - y = 4$, giving $P = (4, 0)$.

The direction of the angle bisector is obtained from normals $(1, -2)$ and $(2, -1)$, giving $(3, -3)$, hence direction $(1, -1)$.

So the bisector through $P$ is $x + y - 4 = 0$.

A direction vector is $(1, -1)$ with magnitude $\sqrt{2}$, so unit vector is $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

After moving 2 km, the displacement is $\left(\sqrt{2}, -\sqrt{2}\right)$, so the point reached is $Q = \left(4 + \sqrt{2}, -\sqrt{2}\right)$.

The river bank is perpendicular to the path, so its direction is $(1, 1)$.

Through $Q$, its equation is $x - y - 4 - 2\sqrt{2} = 0$.

Thus, the river bank is $x - y - 4 - 2\sqrt{2} = 0$ and the point of contact is $\left(4 + \sqrt{2}, -\sqrt{2}\right)$.

334. The sides of the rhombus are parallel to $y = 2x + 3$ and $y = 7x + 2$, so their slopes are 2 and 7.

Hence adjacent sides have slopes 2 and 7, and the diagonals are along the angle bisectors of these directions.

The diagonals intersect at $(1, 2)$, which is the midpoint of both diagonals.

So if a vertex is $A = (0, y)$ (since it lies on the y-axis), its opposite vertex $C$ satisfies midpoint condition:

$\frac{0+x_C}{2} = 1$ and $\frac{y+y_C}{2} = 2$. So $x_C = 2$ and $y_C = 4 - y$.

Thus $C = (2, 4 - y)$. Now slopes of sides must match 2 and 7.

Take adjacent vertex $B$ from $A$ such that $AB$ has slope 2 or 7.

**Case I:** slope of $AB = 2$ $\frac{y_B - y}{x_B - 0} = 2$

Since rhombus is symmetric about diagonals, solving consistently gives: $y = 1$

**Case II:** slope of $AB = 7$ Similarly gives: $y = 3$

So possible vertices on y-axis are: $(0, 1)$ and $(0, 3)$.

335. Let two mutually perpendicular lines be $OX$ and $OY$. $AB$ be the variable line segment of constant length $l$ whose ends $A$ and $B$ move on the lines $OX$ and $OY$ respectively.

Let $OA = a$ and $OB = b$ then $A \equiv (a, 0)$ and $B \equiv (0, b)$.

Let $P$ divide the line segment in the ratio of $1 : 2$. Let $P \equiv (\alpha, \beta)$, then

$\alpha = \frac{2a}{3}$ and $\beta = \frac{b}{3}$. Also, $l^2 = OA^2 + OB^2 \Rightarrow a^2 + b^2 = l^2$

$9\alpha^2 + 36\beta^2 = 4l^2$. Thus, locus of $P$ is $9x^2 + 36y^2 = 4l^2$.

336. Let the line cut the axes at $A = (p \sec\alpha, 0)$ and $B = (0, p\csc\alpha)$. Let $P(h, k)$ be the middle point then $h = \frac{p}{2}\sec\alpha, k = \frac{p}{2}\csc\alpha$.

Thus, $\cos\alpha = \frac{p}{2h}$ and $\sin\alpha = \frac{p}{2k}$

$\cos^2\alpha + \sin^2\alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} \Rightarrow \frac{1}{h^2} + \frac{1}{k^2} = \frac{4}{p^2}$

Hence, lcosu of the point $P$ is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$.

337. The point of intersection of the given lines is given by $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$.

Equation of line passing through this point is given by $y - \frac{ab}{a+b} = m\left(x - \frac{ab}{a+b}\right)$.

$\Rightarrow A = \left(\frac{ab(m-1)}{m(a+b)}, 0\right)$ and $B = \left(0, \frac{ab(1-m)}{a+b}\right)$.

Let $P(\alpha, \beta)$ be the mid-point of $AB$. We have to find its locus i.e. eliminate $m$.

$\alpha = -\frac{ab(1-m)}{2m(a+b)}$ and $\beta = \frac{ab(1-m)}{2(a+b)}$.

$\frac{\alpha}{\beta} = -\frac{1}{m} \Rightarrow m = -\frac{\beta}{\alpha} \Rightarrow 2\alpha\beta(a+b) = ab(\alpha+\beta) \Rightarrow 2xy(a+b) = ab(x+y)$.

338. Equation of any line perpendicular to the given equation passing through the origin is given by $\frac{x}{b} - \frac{y}{b} = 0$

Let foot of the perpendicular from the origin to the given line is intersection of the two lines. Let it be $P(\alpha, \beta)$, then

$\frac{\alpha}{a} + \frac{\beta}{b} = 1$ and $\frac{\alpha}{a} - \frac{\beta}{b} = 0$

Squaring and adding $\alpha^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) + \beta^2\left(\frac{1}{b^2} + \frac{1}{a^2}\right) = 1 \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$.

Here $c$ is a constant and $a, b$ are parameters $\Rightarrow (\alpha^2 + \beta^2).\frac{1}{c^2} = 1$.

Hence, the locus of $P(\alpha, \beta)$ is $x^2 + y^2 = c^2$.

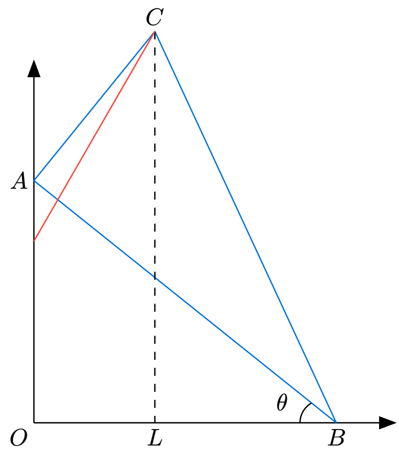
339. Equation of a straight line passing through $(h, k)$ is $y - k = m(x - h)$.

Equation of the straight line perpendicular to the above line passing through origin is $y = -\frac{1}{m}x$.

Let $P(\alpha, \beta)$ be the foot of the perpendicular from the origin to first line. Clearly, $P(\alpha, \beta)$ will be point of intersection of the two lines. Since $P$ lies on both the lines, therefore,

$\beta - k = m(\alpha - x)$ and $\beta = -\frac{1}{m}\alpha \Rightarrow m = -\frac{\alpha}{\beta} \Rightarrow y - k = -\frac{x}{y}(x - h) \Rightarrow x^2 + y^2 = hx + ky$.

340.




Let $OA = c$, then $A = (0, c)$, where $c$ is a constant. Let $C = (\alpha, \beta)$, then $OL = \alpha = OB - BL = c\cot\theta - BC\cos(\theta + 60^\circ)$

$= c\cot\theta - BC(\cos\theta\cos 60^\circ - \sin\theta\sin 60^\circ) = c\cot\theta - \frac{AB}{2}\cos\theta + \frac{AB}{2}\sqrt{3}\sin\theta$

$= c\cot\theta - \frac{1}{2}c\csc\theta\cos\theta + \frac{1}{2}c\csc\theta\sin\theta\sqrt{3}$

$= c\cot\theta - \frac{c}{2}\cot\theta + \frac{\sqrt{3}c}{2} = \frac{c}{2}\left(\cot\theta + \sqrt{3}\right)$

and $\beta = CL = CB\sin(\theta + 60^\circ) = AB\left(\frac{1}{2}\sin\theta + \cos\theta\frac{\sqrt{3}}{2}\right) = \frac{c\csc\theta}{2}\left[\sin\theta + \sqrt{3}\cos\theta\right]$

$= \frac{c}{2} + \frac{\sqrt{3}c}{2}\cot\theta$

Thus, we get $\beta = \sqrt{3}\alpha - c$

Thus, locus of $C$ is $y = \sqrt{3}x - c$, which is a straight line.

341. Let $P(x, y)$. $\Rightarrow \frac{|y-2x+1|}{\sqrt{5}} = \sqrt{x^2 + y^2}$

$\Rightarrow (y - 2x + 1)^2 = 5(x^2 + y^2)$

$\Rightarrow y^2 + 4x^2 + 1 - 4xy + 2y - 4x = 5x^2 + 5y^2$

$\Rightarrow x^2 + 4y^2 + 4xy + 4x - 2y - 1 = 0$

For $y = 2x$: $25x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{5}$

Points: $Q\left(\frac{1}{5}, \frac{2}{5}\right)$ and $R\left(-\frac{1}{5}, -\frac{2}{5}\right)$

Midpoint: $\left(\frac{\frac{1}{5}-\frac{1}{5}}{2}, \frac{\frac{2}{5}-\frac{2}{5}}{2}\right) = (0, 0)$.

342. Equation of any line through origin can be written as $y = mx$.

Solving it with the two equations we have $A = \left(\frac{2}{m+2}, \frac{2m}{m+2}\right)$ and $B = \left(\frac{2}{2m-1}, \frac{2m}{2m-1}\right)$.

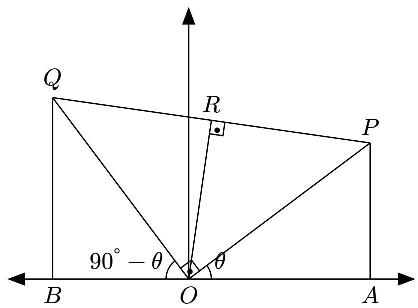
Let the coordinate of the mid-point be $(\alpha, \beta)$. Then $\alpha = \frac{2m+1}{(2m-1)(m+2)}$ and $\beta = \frac{m(3m+1)}{(2m-1)(m+2)}$

$\Rightarrow \frac{\alpha}{\beta} = \frac{1}{m} \Rightarrow m = \frac{\beta}{\alpha}$

Thus, $\alpha = \frac{(3\beta+\alpha)\alpha}{(2\beta-\alpha)+2\alpha}$

Thus, locus is $2x^2 - 3xy - 2y^2 + x + 3y = 0$.

343.




Take $A(h, 0)$ and $B(-h, 0)$.

Let $O(x_1, y_1)$ be a fixed point.

Let $PQ$ be a line through $P(a, b)$ and $Q(c, d)$ such that $OP \perp OQ$.

So: $(a - x_1)(c - x_1) + (b - y_1)(d - y_1) = 0$

Equation of line $PQ$ is: $(y - b) = m(x - a)$ where $m = \frac{d-b}{c-a}$.

Foot of perpendicular from $O(x_1, y_1)$ to $PQ$ is: $R(x, y)$ satisfying: $(x - x_1) + m(y - y_1) = 0$

Also $R$ lies on $PQ$, so: $(y - b) = m(x - a)$

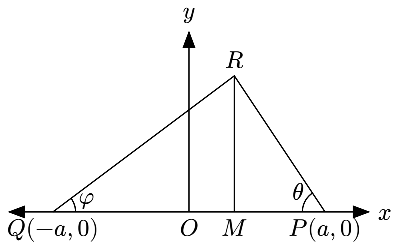
Eliminating $m$ gives: $(x - x_1)(c - a) + (y - y_1)(d - b) = 0$

Using $OP \perp OQ$ condition and simplifying yields: $(x - h)(x + h) + y^2 = 0$

Hence: $x^2 + y^2 = h^2$

This is a circle with center $(0, 0)$ and radius $h$.

Therefore the locus of $R$ is the circle with diameter $AB$.

344. 


Let $PQ$ be taken as $x$-axis and its middle point is $O$, the origin and $OY$ as $y$-axis.

Let $PQ = 2a$ then $P = (a, 0)$ and $Q = (-a, 0)$. Let $R = (h, k)$. Also let $\angle RPQ = \theta$ and $\angle RQP = \varphi$

Given that $\theta - \varphi = 2\alpha$. Let $RM$ be perpendicular to $x$-axis.

Here $\theta$ and $\varphi$ are variables and $a$ and $\alpha$ are constants.

$\tan(\theta - \alpha) = \tan 2\alpha$. $\tan\theta = \frac{k}{a-h}$ and tan phi = k/(a + h)

Putting these values in $\tan(\theta - \alpha)$ yields

$h^2 - k^2 + 2hk\cot 2\alpha - a^2 = 0$

Hence, locus of $R$ is $x^2 - y^2 + 2xy\cot 2\alpha - a^2 = 0$.

345. Equation of the line which passes through the point of intersection of the given lines is given by

$x + 2y - 1 + k(2x - y - 1) = 0$, where $k$ is a parameter.

This line cuts $x$ and $y$ axes at $A = \left(\frac{k+1}{2k+1}, 0\right)$ and $B = \left(0, \frac{k+1}{2-k}\right)$ respectively.

Let $P(\alpha, \beta)$ be the mid-point of $AB$, then $\alpha = \left(\frac{k+1}{2(2k+1)}, \frac{k+1}{2(2-k)}\right)$

$\frac{\alpha}{\beta} = (2-k)(2k+1) \Rightarrow k = \frac{2\beta-\alpha}{2\alpha+\beta}$

Putting $k$ back in $\alpha$ we get $10\alpha\beta = 3\beta + \alpha \Rightarrow 10xy = x + 3y$.

346. Let the variable line through $O$ make an angle $\theta$ with the positive direction of $x$ -axis. Any point on this line will be $(r\cos\theta + r\sin\theta)$.

Let the fixed lines be $y = m_1x + c_1$ and $y = m_2x + c_2$.

Let $OR = r_1, OS = r_2$ and $OP = r_3$, then according to question $\frac{m+n}{r_3} = \frac{m}{r_1} + \frac{n}{r_2}$

Since $A$ and $B$ lies on the fixed lines, therefore, $r_1\sin\theta = m_1(r_1\cos\theta) + c_1$ and $r_2\sin\theta = m_2(r_2\cos\theta) + c_2$

Let $P = (\alpha, \beta)$, then $\alpha = r_3\cos\theta, \beta = r_3\sin\theta$

Thus, $\frac{m+n}{r_3} = m.\frac{\sin\theta - m_1\cos\theta}{c_1} + m_2.\frac{\sin\theta - m_2\cos\theta}{c_2}$

$\Rightarrow m + n = \frac{m}{c_1}(\beta - m_1\alpha) + \frac{m}{c_2}(\beta - m_2\alpha)$

The locus of $P$ is $m + n = \left(\frac{m}{c_1} + \frac{n}{c_2}\right)y - \left(\frac{mm_1}{c_1} + \frac{mm_2}{c_2}\right)x$.

$\Rightarrow y - m_1x + c_1 + \frac{n}{m}.\frac{c_1}{c_2}(y_2 - m_2x - c_2) = 0$

Clearly locus of $P$ is a straight line passing through the point of intersection of the two fixed lines.

347. We take $O$ as the originl Let the $n$ lines be represented by $y = m_rx + c_r$, where $r = 1, 2, 3, ..., n$.

Let $OA$ make an angle $\theta$ with the $x$-axis and cut the given $n$ lines at $n$ diferent points.

Let $OR_1 = r_1, OR_2 = r_2, ..., OR_n = r_n, OR = r \Rightarrow R(\alpha, \beta) = (r\cos\theta, r\sin\theta)$

$y = m_nx + c_n \Rightarrow r_n = \frac{c_n}{\sin\theta - m_n\cos\theta}$

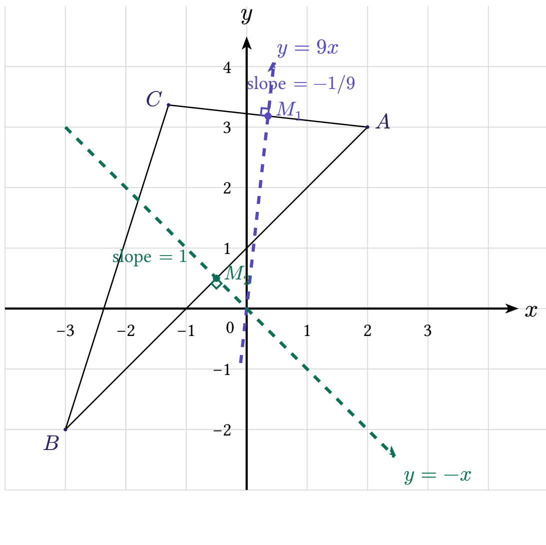
$\Rightarrow \frac{n}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} + \dots + \frac{1}{OR_n}$

$\Rightarrow \frac{n}{r} = \frac{\sin\theta - m_1\cos\theta}{c_1} + \dots + \frac{\sin\theta - m_n\cos\theta}{c_n}$

$\Rightarrow n = \frac{\beta - m_1\alpha}{c_1} + \dots + \frac{\beta - m_n\alpha}{c_n} = \left(\frac{1}{c_1} + \frac{1}{c_2} + \dots + \frac{1}{c_n}\right)\beta - \left(\frac{m_1}{c_1} + \dots + \frac{m_n}{c_n}\right)\alpha$

Hence, locus of $R$ is a straight line.

348.




Let the base of the triangle $ABC$ be $BC$ which passes through a fixed point $P(f, g)$. Let $B = (x_1, y_1)$ and $C = (x_2, y_2)$

Since $P, B, C$ are collinear, therefore,

$$\begin{vmatrix} f & g & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \Rightarrow f(y_1 - y_2) - g(x_1 - x_2) + (x_1y_2 - x_2y_1) = 0$$

Let $A = (\alpha, \beta)$. We have to find the locus of the point $A$.

Given lines are $y^2 - 8xy - 9x^2 = 0 \Rightarrow y - 9x = 0$ and $x + y = 0$.

Let these lines be perpendicular bisectors of $AB$ and $AC$. Since $AB$ is perpendicular to line $y - 9x = 0$, therefore,

$\frac{\beta - y_1}{\alpha - x_1} \times 9 = -1 \Rightarrow x_1 + 9y_1 = \alpha + 9\beta$

Since mid-point $\left(\frac{\alpha + x_1}{2}, \frac{\beta + y_1}{2}\right)$ of $AB$ lies on $y - 9x = 0$, therefore,

$(\beta + x_1) - 9(\beta + y_1) = 0 \Rightarrow 9x_1 - y_1 = \beta - 9\alpha$

Thus, $x_1 = \frac{9\beta - 40\alpha}{41}, y_1 = \frac{40\beta + 9\alpha}{41}$

Proceeding similarly for $AC$ we get $x_2 = -\beta$ and $y_2 = -\alpha$

$\Rightarrow x_1 - x_2 = \frac{50\beta - 40\alpha}{41}$ and $y_1 - y_2 = \frac{40\beta + 50\alpha}{41}$ and $x_1y_2 - x_2y_1 = -\frac{1}{41}[\alpha(9\beta - 40\alpha) - \beta(40\beta + 9\alpha)]$

Putting these in first equation we get locus of $A$ as

$4(x^2 + y^2) + (4g + 5f)x + (4f - 5g)y = 0.$

349. Let the coordinates of the vertex be $(h, k)$, and let the lengths of the bases be respectively $l, l_1, l_2, \ldots$, and their equations be respectively

$x \cos \alpha + y \sin \alpha = p, x \cos \beta + y \sin \beta = p_1, \ldots$ etc.

The length of the perpendiculars from $(h, k)$ on the bases will be

$|h \cos \alpha + k \sin \beta - p|, |h \cos \beta + k \sin \beta - p_1|, \ldots$ etc.

From the given condition sum of areas is constant. Thus,

$x \sum l \cos \alpha + y \sum l \sin \alpha - \sum lp = k$, which is a straight line.

350. Let $A(\cos t, \sin t)$, $B(\sin t, -\cos t)$, $C(1, 2)$.

Centroid $G(x, y)$ is given by $x = \frac{\cos t + \sin t + 1}{3}$, $y = \frac{\sin t - \cos t + 2}{3}$.

So, $3x - 1 = \cos t + \sin t$, $3y - 2 = \sin t - \cos t$.

Now, $(\cos t + \sin t)^2 + (\sin t - \cos t)^2 = 2(\sin^2 t + \cos^2 t) = 2$.

Hence, $(3x - 1)^2 + (3y - 2)^2 = 2$.

351. Given position: $x = u \cos \alpha * t$ and $y = u \sin \alpha * t - \frac{1}{2}gt^2$

From $x = u \cos \alpha * t$, $t = \frac{x}{u \cos \alpha}$

Substitute into $y$: $y = u \sin \alpha * \left(\frac{x}{u \cos \alpha}\right) - \frac{1}{2}g\left(\frac{x}{u \cos \alpha}\right)^2$

$\Rightarrow y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$

Hence, the locus is: $y = x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}$.

352. Let the line through $(1, 1)$ meet the axes at $A(a, 0)$ and $B(0, b)$.

Equation of line in intercept form is $\frac{x}{a} + \frac{y}{b} = 1$

Since it passes through $(1, 1)$, therefore, $\frac{1}{a} + \frac{1}{b} = 1$

Let midpoint of $AB$ be $M(x, y)$. Then, $x = \frac{a}{2}$, $y = \frac{b}{2} \Rightarrow a = 2x$, $b = 2y$

Substitute into $\frac{1}{a} + \frac{1}{b} = 1$ to get $\frac{1}{2x} + \frac{1}{2y} = 1$

$\Rightarrow \frac{x+y}{2xy} = 1 \Rightarrow x + y = 2xy$

Hence, the locus is $2xy = x + y$.

353. Let the line through $(\alpha, \beta)$ meet the axes at $A(a, 0)$ and $B(0, b)$.

Equation in intercept form: $\frac{x}{a} + \frac{y}{b} = 1$.

Since it passes through $(\alpha, \beta)$: $\frac{\alpha}{a} + \frac{\beta}{b} = 1$.

Let midpoint be $M(x, y)$ with $x = \frac{a}{2}$, $y = \frac{b}{2}$. So $a = 2x$, $b = 2y$.

Substitute to get $\frac{\alpha}{2x} + \frac{\beta}{2y} = 1$.

Hence, $\frac{\alpha}{x} + \frac{\beta}{y} = 2$. Multiplying by $xy$ gives $\alpha y + \beta x = 2xy$.

354. Let the line cut the axes at $A(a, 0)$ and $B(0, b)$.

Then its intercept form is $\frac{x}{a} + \frac{y}{b} = 1$

Given condition is $a + b = k$

Midpoint of $AB$ is $M(x, y)$, so $x = \frac{a}{2}$, $y = \frac{b}{2} \Rightarrow a = 2x$, $b = 2y$

Substitute into $a + b = k$ gives $2x + 2y = k$

Hence, the locus is $x + y = \frac{k}{2}$.

355. Let the line meet the axes at $A(x_1, 0)$ and $B(0, y_1)$.

Since $AP = b$ and $PB = a$, the total length is constant, therefore $AB = a + b$

Point $P(x, y)$ divides $AB$ internally in the ratio, thus, $AP : PB = b : a$

Using section formula: $x = \frac{b*0+a*x_1}{a+b} = a\frac{x_1}{a+b}$ $y = \frac{b*y_1+a*0}{a+b} = b\frac{y_1}{a+b}$

So, $x_1 = (a + b)\frac{x}{a}$ and $y_1 = (a + b)\frac{y}{b}$

Since $A$ and $B$ are intercepts of the same line, therefore, $\frac{x_1}{a} + \frac{y_1}{b} = 1$

Substituting values yields $\frac{(a+b)x}{a^2} + \frac{(a+b)y}{b^2} = 1$

Dividing by $(a + b)$ gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

356. Let the variable line cut the axes at $A(a, 0)$ and $B(0, b)$.

Since the line passes through $\left(\frac{6}{5}, \frac{6}{5}\right)$, therefore $\frac{x}{a} + \frac{y}{b} = 1$

Substituting $\left(\frac{6}{5}, \frac{6}{5}\right)$ yields $\frac{6}{5a} + \frac{6}{5b} = 1 \Rightarrow 6(a + b) = 5ab$

Point $P(x, y)$ divides $AB$ internally in ratio $2 : 1$: $\Rightarrow AP : PB = 2 : 1$

Using section formula: $x = \frac{2*0+1*a}{2+1} = \frac{a}{3}$ and $y = \frac{2*b+1*0}{2+1} = 2\frac{b}{3}$

So, $a = 3x$, $b = 3\frac{y}{2}$

Substitute into $6(a + b) = 5ab$ gives us $6\left(3x + 3\frac{y}{2}\right) = 5(3x)\left(3\frac{y}{2}\right)$

Hence, $5xy = 2(2x + y)$.

357. Let the moving line cut the axes at $A(a, 0)$ and $B(0, b)$.

Equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$

Perpendicular distance from origin is given as $p$ is $|ab\frac{|}{\sqrt{a^2+b^2}} = p$

Squaring yields $a^2b^2 = p^2(a^2 + b^2)$

Centroid of triangle $OAB$ is $G(x, y)$: $x = \frac{a}{3}$, $y = \frac{b}{3} \Rightarrow a = 3x$, $b = 3y$

Substitute into distance condition to get $(9x^2)(9y^2) = p^2(9x^2 + 9y^2)$

$81x^2y^2 = 9p^2(x^2 + y^2)$. Hence the locus of centroid is $9x^2y^2 = p^2(x^2 + y^2)$.

358. Let $P(x, y)$ be the moving point. Given $AP \perp BP$.

Slope of $AP = \frac{y-y_1}{x-x_1}$ and slope of $BP = \frac{y-y_2}{x-x_2}$

Since lines are perpendicular, therefore, $\frac{y-y_1}{x-x_1} * \frac{y-y_2}{x-x_2} = -1$

$\Rightarrow (y - y_1)(y - y_2) = -(x - x_1)(x - x_2) \Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Hence the locus is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

359. Let $P(x, y)$ be the moving point.

Square of distance from $(3, -2)$ is $(x - 3)^2 + (y + 2)^2$

Distance from line $5x - 12y = 13$ is $\frac{|5x-12y-13|}{\sqrt{5^2+(-12)^2}} = \frac{|5x-12y-13|}{13}$

Given condition is $(x - 3)^2 + (y + 2)^2 = \frac{|5x-12y-13|}{13}$

Multiply both sides by 13 gives $13[(x - 3)^2 + (y + 2)^2] = |5x - 12y - 13|$

Squaring both sides yields $169[(x - 3)^2 + (y + 2)^2]^2 = (5x - 12y - 13)^2$.

360. Let $P(x, y)$ be the moving point. Fixed points are $F_1(ae, 0)$ and $F_2(-ae, 0)$.

Given $\sqrt{(x - ae)^2 + y^2} + \sqrt{(x + ae)^2 + y^2} = 2a$

This is the definition of an ellipse with foci $F_1, F_2$ and constant sum $2a$.

Squaring and simplifying gives $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$.

361. Let the line $y = x + c$ intersect $2x + 3y = 5$ at $A(x_1, y_1)$ and $2x + 3y = 8$ at $B(x_2, y_2)$.

For $A$: $y = x + c$. Substituting in $2x + 3y = 5$ gives $2x + 3(x + c) = 5 \Rightarrow 5x + 3c = 5$

$\Rightarrow x_1 = \frac{5-3c}{5}$, $y_1 = x_1 + c = \frac{5+2c}{5}$

For $B$: $2x + 3(x + c) = 8 \Rightarrow 5x + 3c = 8 \Rightarrow x_2 = \frac{8-3c}{5}$, $y_2 = \frac{8+2c}{5}$

Midpoint $M(x, y)$: $x = \frac{x_1+x_2}{2} = \frac{13-6c}{10}$ and $y = \frac{y_1+y_2}{2} = \frac{13+4c}{10}$

Eliminating $c$ From $x = \frac{13-6c}{10} \Rightarrow 6c = 13 - 10x$

Substitute into $y$ yielfs $y = \frac{13+4\frac{13-10x}{6}}{10} \Rightarrow 4x + 6y = 13$.

362. Let the point $A$ be $(a, 0)$ on the $x$-axis and the point $B$ be $(b, 6b)$ on the line $y = 6x$.

The length of $AB$ is $2l$, so using the distance formula we get $(b-a)^2 + (6b)^2 = 4l^2$.

The midpoint of $AB$ is $M(x, y)$, so $x = \frac{a+b}{2}$ and $y = \frac{0+6b}{2}$.

From this we obtain $b = \frac{y}{3}$ and $a = 2x - \frac{y}{3}$.

Substituting these values into the length condition gives $\left(\frac{y}{3} - \left(2x - \frac{y}{3}\right)\right)^2 + 36\left(\frac{y}{3}\right)^2 = 4l^2$.

This simplifies to $\left(2\frac{y}{3} - 2x\right)^2 + 4y^2 = 4l^2$.

Dividing by 4 gives $\left(\frac{y}{3} - x\right)^2 + y^2 = l^2$.

363. Let the variable line through $P(-1, 2)$ meet the x-axis at $A(a, 0)$ and the y-axis at $B(0, b)$.

The equation of the line in intercept form is $\frac{x}{a} + \frac{y}{b} = 1$.

Since it passes through $P(-1, 2)$ we get $-\frac{1}{a} + \frac{2}{b} = 1$.

This gives the relation $-b + 2a = ab$.

Let $Q(x, y)$ lie on $AB$. Then $Q$ divides $AB$ in some ratio $t : 1$.

Using section formula we get $x = \frac{a}{1+t}$ and $y = t\frac{b}{1+t}$.

Hence $a = x(1+t)$ and $b = \frac{y(1+t)}{t}$.

Since $P, A, Q, B$ are such that $PA, PQ, PB$ are in harmonic progression we use $PQ^{-1} = \frac{PA^{-1} + PB^{-1}}{2}$.

This gives the relation $\frac{1}{PQ} = \frac{\frac{1}{PA} + \frac{1}{PB}}{2}$.

Using coordinate distances we obtain after simplification that $y = 2x$.

Hence the locus of $Q$ is the line $y = 2x$.

$\frac{1}{PA} + \frac{1}{PB} = \frac{2}{PQ} \Rightarrow y = 2x$

$-\frac{1}{PA} - \frac{1}{PB} = \frac{2}{PQ} \Rightarrow y = 2x + 8$

$\frac{1}{PA} - \frac{1}{PB} = \frac{2}{PQ} \Rightarrow y = -2x + 4$

$-\frac{1}{PA} + \frac{1}{PB} = \frac{2}{PQ} \Rightarrow y = -2x - 4$

Hence the locus is a rhombus bounded by these four lines excluding the vertices.

364. Let $B(x_1, y_1)$ be a variable point.

The perpendicular bisector of $OB$ passes through the midpoint of $O(0,0)$ and $B(x_1, y_1)$, which is $(\frac{x_1}{2}, \frac{y_1}{2})$, and has slope perpendicular to $OB$.

Equation of perpendicular bisector of $OB$ is $x_1x + y_1y = \frac{x_1^2+y_1^2}{2}$

The midpoint of $AB$, where $A(4,4)$ and $B(x_1, y_1)$, is $\left(\frac{x_1+4}{2}, \frac{y_1+4}{2}\right)$.

Slope of $AB$ is $\frac{y_1-4}{x_1-4}$, so the perpendicular bisector of $AB$ is $(x_1-4)\left(x - \frac{x_1+4}{2}\right) + (y_1-4)\left(y - \frac{y_1+4}{2}\right) = 0$

The point of intersection $P(x,y)$ satisfies both equations.

Eliminating $x_1$ and $y_1$ from the two equations gives a relation between $x$ and $y$.

After simplification, we obtain $(x-2)^2 + (y-2)^2 = 8$.

365. Let $O$ be the origin and let $A(a,0)$ and $B(0,b)$ be fixed points on the axes.

Let $C(c,0)$ and $D(0,d)$ be variable points on the axes.

The given condition is $\frac{1}{c} - \frac{1}{d} = \frac{1}{a} - \frac{1}{b}$.

Let $P(x,y)$ be the point of intersection of the lines $AD$ and $BC$.

The equation of the line $AD$ is $y = d\left(1 - \frac{x}{a}\right)$.

The equation of the line $BC$ is $y = b - \left(\frac{b}{c}\right)x$.

At the intersection point $P(x,y)$ we have $d\left(1 - \frac{x}{a}\right) = b - \left(\frac{b}{c}\right)x$.

This simplifies to $d - \left(\frac{d}{a}\right)x = b - \left(\frac{b}{c}\right)x$.

Rearranging gives $(b-d) = x\left(\left(\frac{b}{c}\right) - \left(\frac{d}{a}\right)\right)$.

Using the given condition $\frac{1}{c} - \frac{1}{d} = \frac{1}{a} - \frac{1}{b}$ we rewrite it as $\frac{d-c}{cd} = \frac{b-a}{ab}$.

This gives $ab(d-c) = cd(b-a)$.

On solving the system and eliminating $c$ and $d$ we obtain $x = y$.

Therefore the locus of the point $P$ is $x - y = 0$.

366. Let $Q = (a,k)$. Since $QR$ is the bisector of $\angle OQA$, therefore,

$\frac{OR}{RA} = \frac{OQ}{QA} = \frac{\sqrt{a^2+k^2}}{k} \therefore R = \left(\frac{a\sqrt{a^2+k^2}}{\sqrt{a^2+k^2}+k}, 0\right)$

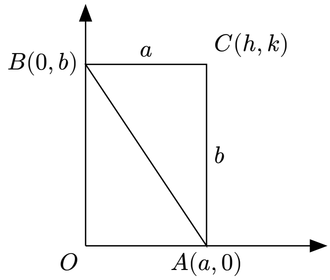
Let $P(\alpha, \beta)$ be the foot of the perpendicular from $R$ to $OQ$.

Now $Q, P, Q$ are collinear. $\therefore \begin{vmatrix} 0 & 0 & 1 \\ \alpha & \beta & 1 \\ a & k & 1 \end{vmatrix} = 0 \Rightarrow k = \frac{a\beta}{\alpha}$

Again $RP \perp QQ \Rightarrow \frac{\beta}{a - \frac{a\sqrt{a^2+k^2}}{\sqrt{a^2+k^2}+k}} \cdot \frac{k}{a} = -1$

Putting the value of $k$ we get locus as $\left(ax - x^2 - y^2\right)^2\left(a^2y^2 + x^2\right) = a^2y^2\left(x^2+y^2\right)^2$.

367.




As shown in the diagram $(h-a)^2+k^2=a^2=CB^2, h^2+(k-b)^2=a^2=CA^2$

We have two equations and two unknowns. Solving thes gives locus of $C(h,k)$ as $bx \pm ay = 0$.

368. Let the isosceles triangle have base $BC$ on the x-axis with $B(-a,0)$ and $C(a,0)$, and vertex $A(0,h)$.

Let $P(x,y)$ be the moving point.

The distance from $P$ to the base $BC$ is $y$, so the square of the distance is $y^2$.

The equation of line $AB$ is $hx+ay-ah=0$, so the distance from $P$ to $AB$ is $\frac{|hx+ay-ah|}{\sqrt{h^2+a^2}}$.

The equation of line $AC$ is $hx-ay+ah=0$, so the distance from $P$ to $AC$ is $\frac{|hx-ay+ah|}{\sqrt{h^2+a^2}}$.

The given condition is $y^2 = \frac{|hx+ay-ah|}{\sqrt{h^2+a^2}} \times \frac{|hx-ay+ah|}{\sqrt{h^2+a^2}}$.

This simplifies to $y^2(h^2+a^2) = (hx+ay-ah)(hx-ay+ah)$.

Expanding the product gives $h^2x^2 - a^2y^2 + a^2h^2$.

Substituting gives $y^2(h^2+a^2) = h^2x^2 - a^2y^2 + a^2h^2$.

Rearranging gives $h^2x^2 + a^2h^2 = y^2(h^2+2a^2)$.

Dividing by $h^2$ gives $x^2 + a^2 - y^2 - \left(2\frac{a^2}{h^2}\right)y^2 = 0$.

Rearranging into standard form gives $x^2 + y^2\left(1+2\frac{a^2}{h^2}\right) = a^2$.

Therefore the locus of $P$ is a circle.

369. Let $O$ be the origin. A variable line through $O$ meets two fixed straight lines at $R$ and $S$.

Let $P$ lie on this line such that $\frac{2}{OP} = \frac{1}{OR} + \frac{1}{OS}$.

Since $O, R, S, P$ are collinear, let $OR = r$, $OS = s$, and $OP = p$.

The given condition is $\frac{2}{p} = \frac{1}{r} + \frac{1}{s}$.

This gives $2 = p\left(\frac{1}{r} + \frac{1}{s}\right)$.

Multiplying by $rs$ gives $2rs = p(r+s)$.

Hence $p = \frac{2rs}{r+s}$.

Let the variable line through $O$ have direction ratio $(1, m)$ so that $R = (r, mr)$, $S = (s, ms)$, and $P = (p, mp)$.

The points $R$ and $S$ lie on two fixed straight lines, so $r$ and $s$ satisfy linear equations depending on $m$.

Eliminating $r$ and $s$ between these two linear relations and the condition $2rs = p(r+s)$ removes the parameter $m$.

The resulting relation between $x$ and $y$ is linear.

Therefore the locus of $P$ is a straight line.

370. The $i$-th terms are $x_i = p + (i-1)a$ and $y_i = q + (i-1)b$.

The mean of the first $n$ terms of an arithmetic sequence is the average of first and last terms.

Hence $\alpha = \frac{x_1 + x_n}{2}$.

Substituting gives $\alpha = \frac{p + (p + (n-1)a)}{2}$.

So $\alpha = p + (n-1)\frac{a}{2}$.

Similarly $\beta = \frac{y_1 + y_n}{2} = q + (n-1)\frac{b}{2}$.

Eliminating $n$ from both equations gives $\frac{\alpha - p}{a} = \frac{\beta - q}{b}$.

Hence, $b(\alpha - p) = a(\beta - q)$ i.e. the locus is $bx - ay + qa - bp = 0$.

# 4 Answers of Pair of Straight Lines

1. The joint equation is given by $(x-4y+2)(x-y-1)=0 \Rightarrow x^2-5xy+4y^2+x+2y-2=0$.

2. Let $y=mx$ be the straight line passing through origin and making an angle $\alpha$ with $y+x=0$.

   Then $\tan\alpha = \frac{|m+1|}{1+m} \Rightarrow m^2(1-\tan^2\alpha)+2m(1+\tan^2\alpha)+(1-\tan^2\alpha)=0$

   $\Rightarrow m_1+m_2=-2\sec 2\alpha, m_1m_2=1$.

   Thus, $(y-m_1x)(y-m_2x)=x^2+2xy\sec 2\alpha+y^2=0$.

3. Since the pair of lines are perpendicular and form an isosceles triangle with $2x+3y=6$ the pair will make $45^\circ$ with it. Let $m$ be the slope of pair of straight lines. Then

   $\tan 45^\circ = 1 = \left|\frac{m-(-\frac{2}{3})}{1}+\frac{2m}{3}\right| \Rightarrow m=\frac{1}{5},-5$. So the lines are $5y-x=0$ and $5x+y=0$ as they pass through origin.

   Solving the three lines pairwaise we obtain vertices as $(0,0), \left(\frac{20}{13},\frac{6}{13}\right), \left(-\frac{6}{13},\frac{30}{13}\right)$.

   Thus, $\Delta = \frac{936}{169}$.

4. The combined equation is given by $(2x-y-3)(3x-y+4)=6x^2-5xy+y^2-x-y-12=0$.

5. Lines parallel to $x-2y=5$ and $x=3y-4$ are given by $x-2y=c$ and $x-3y=k$.

   Given that both pass through $(1,2)$, thus, $c=-3$ and $k=-5$. So the lines are $x-2y+3=0$ and $x-3y+5=0$.

   Thus, combined equation is $(x-2y+3)(x-3y+5)=x^2-5xy+6x^2+8x-19y+15=0$.

6. The bisectors of angle between coordinates axes is given by $x-y=0$ and $x+y=0$(verify).

   The combined equation will be therefore $x^2-y^2=0$.

7. Comparing given equation with general equation in second degree we have $a=8, b=2, c=15, 2h=8, 2g=26, 2f=13$

   Now $abc+2fgh-af^2-bg^2-ch^2=0$. Hence, the given equation represents pair of straight lines.

8. Comparing the given equation with the general equation of second degree we get $a=6, h=k, b=12, g=11, f=\frac{31}{2}, c=20$

   Putting these values in $abc+2fgh-af^2-bg^2-ch^2=0$ gives us

   $40k^2-682k+2907=0 \Rightarrow k=\frac{682\pm 2}{80}=\frac{171}{20},\frac{17}{2}$.

9. Comparing the given equation with the general equation of second degree we get $a = 10, 2h = -11, b = -6, g = -6, f = -\frac{1}{2}, c = 2$

   Putting these values in $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ gives us

   $0 = 0$. Hence, the give equation represents pair of straight lines.

10. Comparing the given equation with the general equation of second degree we get $a = 2, 2h = -15, b = -17, g = 2, f = \frac{23}{2}, c = -6$

    Putting these values in $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ gives us

    $0 = 0$. Hence, the give equation represents pair of straight lines.

11. Comparing the given equation with the general equation of second degree we get $a = m, 2h = -5, b = -6, 2g = 14, 2f = 5, c = 4$

    Putting these values in $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ gives us

    $m = 6$. We see that $a + b = 0$, hence, the lines are perpendicular to each other.

12. Comparing the given equation with the general equation of second degree we get $a = 1, 2h = m, b = -2, 2g = 0, 2f = 3, c = -1$

    Putting these values in $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ gives us $m = \pm 1$.

13. Comparing the given equation with the general equation of second degree we get $a = 12, 2h = -10, b = 2, 2g = 11, 2f = -5, c = m$

    Putting these values in $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ gives us $m = 2$.

14. Comparing the given equation with the general equation of second degree we get $a = 6, 2h = 5, b = -4, 2g = 7, 2f = m, c = 2$

    Putting these values in $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ gives us $m = 2$ or $\frac{23}{6}$.

15. $a = 4, 2h = 24, b = 11$ from the given equation.

    Let $\theta$ be the angle between the pair of straight lines. Then $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{4}{3}$, so the acute angle between the lines is $\tan^{-1}\frac{4}{3}$.

16. Given equation is $(x^2 + y^2)\sin^2\alpha = (x\cos\beta - y\sin\beta)^2$.

    $\Rightarrow (\cos^2\beta - \sin^2\alpha)x^2 - 2\sin\beta\cos\beta xy + (\sin^2\beta - \sin^2\alpha)$y^2 = 0

    $\Rightarrow a = \cos^2\beta - \sin^2\alpha, 2h = -\sin 2\beta, b = \sin^2\beta - \sin^2\alpha$

    Let $\theta$ be the angle between the lines. Then $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} = \pm\tan 2\alpha$

    Thus, angle between the straight lines is $2\alpha$.

17. Comparing with the general equation, we have

    $a = 1, 2h = -5, b = 4$. If $\theta$ is the angle between the pair of straight lines, then $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{3}{5}$

Thus, acute angle between the lines is $\tan^{-1}\frac{3}{5}$.

18. We can write the given equation as $y^2 - \frac{2}{\sin\theta\cos\theta}xy + \left(\frac{\sin^4\theta+\cos^4\theta}{\sin^2\theta\cos^2\theta}\right)x^2 = 0$

Let the straight lines by $y - x\tan\alpha = 0$ and $y - x\tan\beta = 0$, then

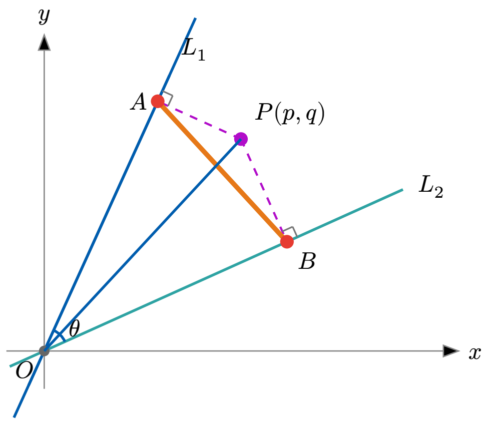
$\tan\alpha + \tan\beta = \frac{2}{\sin\theta\cos\theta}$, and $\tan\alpha\tan\beta = \frac{\sin^4\theta+\cos^4\theta}{\sin^2\theta\cos^2\theta}$

$(\tan\alpha - \tan\beta)^2 = (\tan\alpha + \tan\beta)^2 - 4\tan\alpha\tan\beta$

$= \frac{4}{\sin^2\theta\cos^2\theta} - \frac{4(\sin^4\theta+\cos^4\theta)}{\sin^2\theta\cos^2\theta} = \frac{4}{\sin^2\theta\cos^2\theta}(1 - \sin^2\theta - \cos^4\theta) = 4$

$\Rightarrow \tan\alpha - \tan\beta = 2.$

19.




Let $\theta$ be the angle between the pair of lines given by the equaiton $ax^2 + 2hxy + by^2 = 0$. Let $A, B$ be the foot of perpendiculars from $P(p, q)$ on the two lines.

If we draw a circle with $OP$ as diameter then both $A$ and $B$ will be on tihs circle because diameter $OP$ will make right angle at any point on the circumference.

$\therefore \angle OAB = \angle OPB = \alpha$(let)

From $\triangle\, OAB$, we have $\frac{OB}{\sin\alpha} = \frac{AB}{\sin\theta}$, and from $\triangle\, OPB$, we have $\frac{OP}{\sin 90^\circ} = \frac{OB}{\sin\alpha} = \frac{AB}{\sin\theta}$

$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} = \frac{2\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}} \Rightarrow AB = OP\sin\theta = \frac{2\sqrt{p^2+q^2}\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}}.$

20. From previous question $AB = \frac{2\sqrt{p^2+q^2}\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}}$

$\therefore 2k = \frac{2\sqrt{p^2+q^2}\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}}$

Hence, locus of point $P(p, q)$ is $(x^2 + y^2)(h^2 - ab) = k^2\left[(a - b)^2 + 4h^2\right]$.

21. Comparing with the general equation yields $a = 1, b = -1$. Since $a + b = 0$, the lines will be perpendicular to each other.

22. Like previour problem $a + b = 0$, the lines will be perpendicular to each other.

23. Comparing with the general equation yields $a = 1, b = 1, h = -p$.

If $\theta$ is the angle between lines then $\tan\theta = \frac{2\sqrt{p^2-1}}{2} = \sqrt{p^2-1}$

$\Rightarrow \theta = \tan^{-1}\sqrt{p^2-1}$.

24. Comparing with the general equation yields $a = 1, b = 1, h = -\sec\theta$.

    If $\alpha$ is the angle between the lines then $\tan\alpha = \frac{2\sqrt{\sec^2\theta-1}}{2} = \tan\theta$

    Thus, the angle made by the pair of straight lines with one another is $\theta$.

25. The given equation is $\left((x^2+y^2)\sin^2\alpha = (x\cos\alpha - y\sin^2\alpha)\right)$.

    Expanding the right-hand side gives $(x^2\cos^2\alpha - 2xy\sin\alpha\cos\alpha + y^2\sin^2\alpha)$.

    So the equation becomes $x^2\sin^2\alpha + y^2\sin^2\alpha = x^2\cos^2\alpha - 2xy\sin\alpha\cos\alpha + y^2\sin^2\alpha$.

    Canceling $y^2\sin^2\alpha$ from both sides gives $x^2\sin^2\alpha - \cos^2\alpha + 2xy\sin\alpha\cos\alpha = 0$.

    Using identities $\sin^2\alpha - \cos^2\alpha = -\cos 2\alpha$ and $2\sin\alpha\cos\alpha = \sin 2\alpha$, we get $-x^2\cos 2\alpha + xy\sin 2\alpha = 0$.

    Factoring gives $x(-x\cos 2\alpha + y\sin 2\alpha) = 0$.

    So the two lines are $x = 0$ and $y\sin 2\alpha - x\cos 2\alpha = 0$.

    The first line has slope undefined, and the second line has slope $m = \frac{\cos 2\alpha}{\sin 2\alpha} = \cot 2\alpha$.

    Using the angle formula between lines, the angle between them is $2\alpha$.

26. The given equation is $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$.

    Compare with the general second degree form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

    We identify $a = 6$, $2h = -5$ so $h = -\frac{5}{2}$, and $b = -6$.

    To check if it represents a pair of straight lines, we use the condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

    Here $g = 7$, $f = \frac{5}{2}$, and $c = 4$.

    Substituting gives $6 * (-6) * 4 + 2 * \left(\frac{5}{2}\right) * 7 * \left(-\frac{5}{2}\right) - 6 * \left(\frac{5}{2}\right)^2 - (-6) * 7^2 - 4 * \left(-\frac{5}{2}\right)^2$.

    This simplifies to $-144 - \frac{175}{2} - \frac{150}{4} + 294 - \frac{100}{4} = 0$, so the condition is satisfied.

    Hence the equation represents a pair of straight lines.

    Now for perpendicularity, we use the condition $a + b = 0$.

    Here $a + b = 6 + (-6) = 0$, so the lines are perpendicular.

27. We identify $a = 16$, $2h = 24$ so $h = 12$, and $b = 9$.

    First check if it represents a pair of straight lines using the condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

    Here $g = 20$, $f = 15$, and $c = -75$.

Substituting gives $16 * 9 * (-75) + 2 * 15 * 20 * 12 - 16 * 15^2 - 9 * 20^2 - (-75) * 12^2$.

This simplifies to $-10800 + 7200 - 3600 - 3600 + 10800 = 0$, so the condition is satisfied.

Hence it represents a pair of straight lines.

Now check for parallel lines using the condition $h^2 = ab$.

Here $h^2 = 12^2 = 144$ and $ab = 16 * 9 = 144$.

Since $h^2 = ab$, the two lines are parallel.

28. $a = 1, h = -\frac{5}{2}, b = 4, g = \frac{1}{2}, f = 1$, and $c = -2$.

    Check the condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

    Substituting gives $1 * 4 * (-2) + 2 * 1 * \left(\frac{1}{2}\right) * \left(-\frac{5}{2}\right) - 1 * 1^2 - 4 * \left(\frac{1}{2}\right)^2 - (-2) * \left(-\frac{5}{2}\right)^2$.

    This simplifies to $-8 - \frac{5}{2} - 1 - 1 - \frac{25}{2} = 0$, so the condition is satisfied.

    Hence it represents a pair of straight lines.

    Now the angle between the lines is given by $\tan\theta = 2 * \frac{\sqrt{h^2 - a * b}}{a+b}$.

    Substitute $h^2 = \frac{25}{4}$ and $a * b = 4$. So $h^2 - a * b = \frac{25}{4} - \frac{16}{4} = \frac{9}{4}$.

    Then $\tan\theta = \frac{2 * \frac{3}{2}}{5} = \frac{3}{5} \Rightarrow \theta = \tan^{-1}\frac{3}{5}$.

29. $a = 12$, $h = \frac{7}{2}$, $b = -p$, $g = -9$, $f = \frac{q}{2}$, $c = 6$.

    For perpendicular lines: $a + b = 0$ gives $12 - p = 0$ so $p = 12$.

    Using $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ gives $q = 1$.

30. $2x^2 + 3xy - 2y^2 = 0 \Rightarrow (2x - y)(x + 2y) = 0$, which gives two straight lines through origin and perpendicular to one another.

31. Given equation can be written as $2x^2 - (y - 9)x - (y^2 + 3y - 10) = 0$

    Solving for $x$ gives $4x = y - 9 \pm (3y + 1) \Rightarrow x - y + 2 = 0$ and $2x + y + 5 = 0$.

32. We can write the given equation as a quadratic equation in $x$ as follows:

    $2x^2 + (5y + 6)x + (3y^2 + 7y + 4) = 0$

    Solving for $x$ yields $4x = -(5y + 6) \pm (y + 2) \Rightarrow x + y + 1 = 0$ and $2x + 3y + 4 = 0$.

    Point of intersection is $(1, -2)$ and angle between them would be $\tan^{-1}\frac{1}{5}$.

33. We can write the given equation as a quadratic equation in $x$ as follows:

    $8x^2 + (8y + 26)x + 2y^2 + 13y + 15 = 0$

    Solving for $x$ yields $16x = -(8y + 26) \pm 14 \Rightarrow 4x + 2y + 3 = 0$ and $2x + y + 5 = 0$.

    Thus, we have a pair of parallel straight lines.

Perpendicular distance between them is $\frac{|5-\frac{3}{2}|}{\sqrt{2^2+1}} = \frac{7}{2\sqrt{5}}$.

34. We can write the given equation as a quadratic equation in $x$ as follows:
$x^2 - (5y-1)x + (4y^2 + 2y - 2) = 0$

Solving for $x$ yields $2x = (5y-1) \pm 3(y-1) \Rightarrow x - 4y + 2 = 0$ and $x - y - 1 = 0$

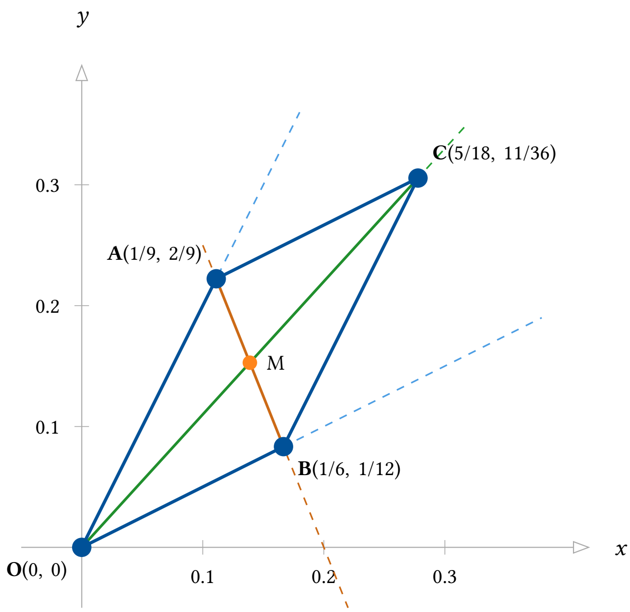
Lines parallel to these lines and passing through $(1, 1)$ are given by

$x - 4y - (1 - 4.2) = 0$ and $x - y - (1 - 1) = 0$ i.e. $x - 4y + 3 = 0$ and $x = y$.

The combined equation is $(x - 4y + 3)(x - y) = 0 \Rightarrow x^2 - 5xy + 4y^2 + 3x - 3y = 0$

Thus, angle between lines is $\tan\theta = \left|\frac{2\sqrt{\frac{25}{4}-4}}{1+4}\right| = \frac{3}{5} \Rightarrow \theta = \tan^{1-}\frac{3}{5}$.

35.




Given pair of lines is $2x^2 - 5xy + 2y^2 = 0 \Rightarrow (x - 2y)(2x - y) = 0 \Rightarrow x - 2y = 0$ and $2x - y = 0$ are the two lines represenetd by it.

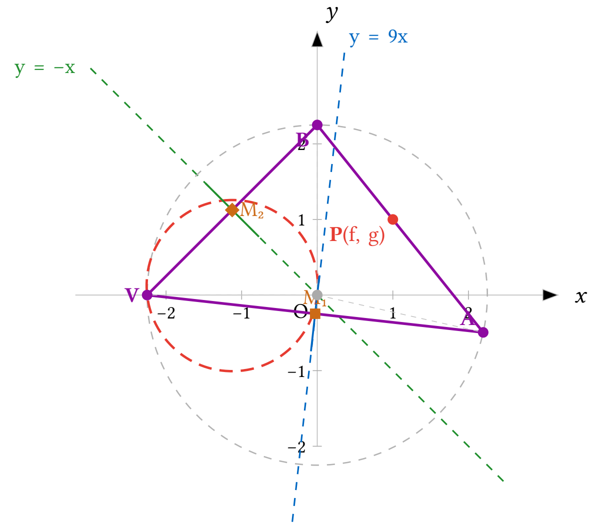
Let these sides are $OA$ and $OB$. Let the given diagonal be $AB$ i.e. $5x + 2y = 1$.

Solving with the parallel sides we get $B$ as $\left(\frac{1}{6}, \frac{1}{12}\right)$ and $A$ as $\left(\frac{1}{9}, \frac{2}{9}\right)$. Thus, mid-point is $\left(\frac{5}{36}, \frac{11}{72}\right)$.

Thus, equation of other diagonal would be $11x - 10y = 0$ as it passes through $H$ and $O$.

Thus, $\Delta_{OAB} = \frac{1}{72}$ and area of parallelogram is $\frac{1}{36}$.

36.




The given pair of lines is $y^2 - 8xy - 9x^2 = 0 \Rightarrow y - 9x = 0$ and $x + y = 0$.

Let the vertex be $(\alpha, \beta)$ then $9y + x - (\alpha + 9\beta) = 0$ and $x - y + (\alpha - \beta) = 0$ will be the equation of two sides perpendicular to the bisectors.

Let two other vertices be $(x_1, y_1)$ and $(x_2, y_2)$. Then $9y_1 + x_1 = \alpha + 9\beta$ and $x_2 - y_2 = \alpha - \beta$

Also, mid-points will be $\frac{\frac{\alpha + x_1}{2}, (\beta + y_1)}{2}$ and $\frac{\frac{\alpha + x_2}{2}, (\beta + y_2)}{2}$.

Solving we get $(x_1, y_1) = \left(\frac{9\beta - 40\alpha}{41}, \frac{40\beta + 9\alpha}{41}\right)$ and $(x_2, y_2) = (-\beta, -\alpha)$

$P$ is collinear with these two points, therefore,

$$\begin{vmatrix} f & g & 1 \\ -\beta & -\alpha & 1 \\ \frac{9\beta - 40\alpha}{41} & \frac{40\beta + 9\alpha}{41} & 1 \end{vmatrix} = 0$$

Thus, we get locus as $4(x^2 + y^2) + (4g + 5f)x + (4f - 5g)y = 0$.

37. The two pair of straight lines are $y - mx = \pm a\sqrt{1 + m^2}$ and $y - nx = \pm a\sqrt{1 + n^2}$

Clearly the two pairs are parallel to each other. The distance between the lines are $|2a|$ for all pairs.

Thus, the given pair of straight lines form a rhombus.

38. Comparing given equation with the general equation of second degree we have

$a = 0, b = 0, c = c, h = h, g = g, f = f$

Since the equation represents two straight lines, theerfore,

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \Rightarrow c = \frac{2fg}{h}$

Thus, the equation becomes $2hxy + 2gx + 2fy + + \frac{2fg}{h} = 0 \Rightarrow (hx + f)(hy + g) = 0$

$x = -\frac{f}{h}, y = -\frac{g}{h}$, and clearly, these lines will make a rectangle with coordinate axes.

Area of the rectangle is $\left|-\frac{f}{h}\right| \times \left|-\frac{g}{h}\right| = \frac{|fg|}{h^2}$.

39. Assume given line represents a pair of straight lines: $(x + Ay + B)(x + Cy + D) = 0$.

    Expanding, $x^2 + (A + C)xy + ACy^2 + (B + D)x + (AD + BC)y + BD = 0$.

    Comparing coefficients: $A + C = -5$, $AC = 4$, $B + D = 1$, $AD + BC = 2$, $BD = -2$.

    From $AC = 4$ and $A + C = -5$, we get $A = -1, C = -4$.

    Now solve for $B, D$ using $B + D = 1$ and $BD = -2$, giving $B = -1, D = 2$.

    Thus the factorization is $(x - y - 1)(x - 4y + 2) = 0$.

    Hence, the separate equations of the lines are $x - y - 1 = 0$ and $x - 4y + 2 = 0$.

40. The given homogeneous equations represent pairs of straight lines through the origin.

    First equation is $x^2 - 6xy + 3y^2 = 0$. Divide by $x^2$ assuming $x \neq 0$ to get $1 - 6\left(\frac{y}{x}\right) + 3\left(\frac{y}{x}\right)^2 = 0$.

    Let $m = \frac{y}{x}$. Then $3m^2 - 6m + 1 = 0$.

    The slopes are given by $m = \frac{6+-\sqrt{36-12}}{6} = \frac{6+-2\sqrt{6}}{6} = 1 + -\frac{\sqrt{6}}{3}$.

    So the slopes are $m_1 = 1 + \frac{\sqrt{6}}{3}$ and $m_2 = 1 - \frac{\sqrt{6}}{3}$.

    Second equation is $3x^2 + 6xy + y^2 = 0$. Divide by $x^2$ to get $3 + 6\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = 0$.

    Let $m = \frac{y}{x}$. Then $m^2 + 6m + 3 = 0$.

    The slopes are $m = \frac{-6+-\sqrt{36-12}}{2} = \frac{-6+-2\sqrt{6}}{2} = -3 + -\sqrt{6}$.

    So the slopes are $m_3 = -3 + \sqrt{6}$ and $m_4 = -3 - \sqrt{6}$.

    Now check perpendicularity using the condition that the product of slopes is $-1$.

    Compute $m_1 m_3 = \left(1 + \frac{\sqrt{6}}{3}\right)\left(-3 + \sqrt{6}\right) = -3 + \sqrt{6} - \sqrt{6} + 2 = -1$.

    So $m_1$ is perpendicular to $m_3$.

    Compute $m_2 m_4 = \left(1 - \frac{\sqrt{6}}{3}\right)\left(-3 - \sqrt{6}\right) = -3 - \sqrt{6} + \sqrt{6} + 2 = -1$.

    So $m_2$ is perpendicular to $m_4$.

    Thus each line from the first equation is perpendicular to one line from the second equation.

41. Given pairs of lines are $y^2 + xy - 12x^2 = 0$ and $4y^2 - 13xy + 3x^2 = 0$

    $y^2 + xy - 12x^2 = y^2 + 4xy - 3xy - 12x^2 = 0 \Rightarrow (y + 4x)(y - 3x) = 0$

    $4y^2 - 13xy + 3x^2 = 4y^2 - 12xy - xy + 3x^2 = (y - 3x)(4y - x) = 0$

We see that $y-3x=0$ conincides and the other pair i.e. $y+4x=0$ and $4y-x=0$ are perpendicular to one another.

42. $x^2-7xy+12y^2=x^2-3xy-4xy+12y^2=(x-3y)(x-4y)=0$

$12x^2+7xy+y^2=12x^2+3xy+4xy+y^2=(4x+y)(3x+y)=0$

Clearly, the lines are perpendicular to each other.

43. From $x^2-7x+6=0$ we get $(x-1)(x-6)=0$, so the lines are $x=1$ and $x=6$.

From $y^2-14y+40=0$ we get $(y-4)(y-10)=0$, so the lines are $y=4$ and $y=10$.

These four lines form a rectangle with vertices at $(1,4)$, $(1,10)$, $(6,4)$ and $(6,10)$.

The diagonals join opposite vertices.

First diagonal passes through $(1,4)$ and $(6,10)$. Its slope is $\frac{10-4}{6-1}=\frac{6}{5}$.

Using point-slope form $y-4=\left(\frac{6}{5}\right)(x-1)$ which simplifies to $6x-5y+14=0$.

Second diagonal passes through $(1,10)$ and $(6,4)$. Its slope is $\frac{4-10}{6-1}=-\frac{6}{5}$.

Using point-slope form $y-10=\left(-\frac{6}{5}\right)(x-1)$ which simplifies to $6x+5y-56=0$.

44. The equation $3x^2+8xy-3y^2=0$ represents a pair of straight lines through the origin.

Divide by $x^2$ and put $m=\frac{y}{x}$. Then $3+8m-3m^2=0$ which gives $3m^2-8m-3=0$.

Solving, $m=\frac{8+-\sqrt{64+36}}{6}=\frac{8+-10}{6}$. So $m=3$ and $m=-\frac{1}{3}$.

Thus the lines are $y=3x$ and $y=-\frac{x}{3}$.

Now consider $3x^2+8xy-3y^2+2x-4y-1=0$. This represents another pair of lines parallel to the first pair.

Rewrite it as $3x^2+8xy-3y^2+2x-4y-1=(y-3x+1)(3y+x+1)$.

So the lines are $y=3x-1$ and $y=-\frac{x}{3}-\frac{1}{3}$.

Hence we have two pairs of parallel lines $y=3x$ and $y=3x-1$, $y=-\frac{x}{3}$ and $y=-\frac{x}{3}-\frac{1}{3}$.

The slopes satisfy $3*\left(-\frac{1}{3}\right)=-1$, so the adjacent sides are perpendicular. Also the perpendicular distance between each pair is equal.

Distance between $y=3x$ and $y=3x-1$ is $\frac{1}{\sqrt{1+9}}=\frac{1}{\sqrt{10}}$.

Distance between $y=-\frac{x}{3}$ and $y=-\frac{x}{3}-\frac{1}{3}$ is $\frac{\frac{1}{3}}{\sqrt{1+\frac{1}{9}}}=\frac{1}{\sqrt{10}}$.

Thus the four lines form a square.

Vertices are obtained by intersection of adjacent lines

Intersection of $y = 3x$ and $y = -\frac{x}{3}$ gives $(0, 0)$, Intersection of $y = 3x - 1$ and $y = -\frac{x}{3}$ gives $\left(\frac{3}{10}, -\frac{1}{10}\right)$, Intersection of $y = 3x$ and $y = -\frac{x}{3} - \frac{1}{3}$ gives $\left(-\frac{1}{10}, -\frac{3}{10}\right)$, and Intersection of $y = 3x - 1$ and $y = -\frac{x}{3} - \frac{1}{3}$ gives $\left(\frac{1}{5}, -\frac{2}{5}\right)$.

One diagonal joins $(0, 0)$ and $\left(\frac{1}{5}, -\frac{2}{5}\right)$. Slope is $-2$ so equation is $y = -2x$ or $2x + y = 0$.

Other diagonal joins $\left(\frac{3}{10}, -\frac{1}{10}\right)$ and $\left(-\frac{1}{10}, -\frac{3}{10}\right)$. Slope is $\frac{1}{2}$. Using point-slope form $y + \frac{1}{10} = \left(\frac{1}{2}\right)\left(x - \frac{3}{10}\right)$ which simplifies to $x - 2y - \frac{1}{2} = 0$.

45. $2x^2 - 5xy - 3y^2 - 2x + 6y = (2x^2 - 5xy - 3y^2) + (-2x + 6y)$.

    Factor the quadratic part $2x^2 - 5xy - 3y^2 = (2x + y)(x - 3y)$.

    So the equation becomes $(2x + y)(x - 3y) - 2(x - 3y)$.

    Factor out $(x - 3y) = (x - 3y)(2x + y - 2)$.

    Hence the given equation reduces to $(x - 3y)(2x + y - 2) = 0$.

    Therefore it represents two straight lines $x - 3y = 0$ and $2x + y - 2 = 0$.

    From $x - 3y = 0$ we get $x = 3y$. Substitute into $2x + y - 2 = 0 \Rightarrow 2(3y) + y - 2 = 0 \Rightarrow 7y = 2$ so $y = \frac{2}{7}$. Then $x = 3y = \frac{6}{7}$.

46. Let one line represents by $ax^2 + 2hxy + by^2 = 0$ is $y - mx = 0$. Thus,

    $am^2x^2 + 2hxmx + bm^2x^2 = 0 \Rightarrow a + 2hm + bm^2 = 0$

    Then according to question on of the lines of $a_1x^2 + 2h_1xy + b_1y^2 = 0$ would be $my + x = 0$

    $\Rightarrow a_1(-my)^2 + 2h(-my)y + b_1y^2 = 0 \Rightarrow b_1 - 2h_1m + a_1m^2 = 0$

    From two obtained equation we cross-multiply to get

    $\frac{1}{2(ha_1+h_1b)} = \frac{m}{bb_1-aa_1} = \frac{m^2}{-2(ah_1+b_1h)}$

    Thus, $(bb_1 - a\frac{a_1}{2(ha_1+h_1b)} = -\frac{2(ah_1+b_1h)}{bb_1-aa_1}$

    $\Rightarrow (bb_1 - aa_1)^2 + 4(h_1a + b_1h)(ha_1 + bh_1) = 0.$

47. Consider the equation $ax^2 + 2hxy + by^2 = 0$ which represents two straight lines through the origin.

    Let their slopes be $m_1$ and $m_2$. Put $y = mx$ so that $a + 2hm + bm^2 = 0$ or $bm^2 + 2hm + a = 0$.

    Thus $m_1$ and $m_2$ are the roots of $bm^2 + 2hm + a = 0$.

    So $m_1 + m_2 = -2\frac{h}{b}$ and $m_1m_2 = \frac{a}{b}$.

    Given that one slope is $\lambda$ times the other, let $m_1 = \lambda m_2$.

    Then $m_1 + m_2 = \lambda m_2 + m_2 = (1 + \lambda)m_2$ and $m_1m_2 = \lambda m_2^2$.

    Using $m_1m_2 = \frac{a}{b}$ we get $\lambda m_2^2 = \frac{a}{b}$ so $m_2^2 = \frac{a}{b\lambda}$.

    Now using $m_1 + m_2 = -2\frac{h}{b}$ we get $(1 + \lambda)m_2 = -2\frac{h}{b}$.

$\Rightarrow \left(\frac{1+\lambda}{2h}\right)^2 = \frac{\lambda}{ab}$.

48. Consider the equation $ax^2 + 2hxy + by^2 = 0$ which represents two straight lines through the origin.

    Let their slopes be $m_1$ and $m_2$. Putting $y = mx$, we get $a + 2hm + bm^2 = 0$ or $bm^2 + 2hm + a = 0$.

    Thus $m_1$ and $m_2$ are the roots of this equation, so $m_1 + m_2 = -2\frac{h}{b}$ and $m_1 m_2 = \frac{a}{b}$.

    Given that one slope is the square of the other, let $m_1 = m_2^2$.

    Then $m_1 m_2 = m_2^3 = \frac{a}{b}$ so $m_2^3 = \frac{a}{b}$.

    Also $m_1 + m_2 = m_2^2 + m_2 = -2\frac{h}{b}$.

    Multiply this by $m_2$, $m_2^3 + m_2^2 = \left(-2\frac{h}{b}\right)m_2$.

    Using $m_2^3 = \frac{a}{b}$, we get $\frac{a}{b} + m_2^2 = \left(-2\frac{h}{b}\right)m_2$.

    Multiply by $b$, $a + bm_2^2 = -2hm_2$.

    Now square both sides, $\left(a + bm_2^2\right)^2 = 4h^2 m_2^2$.

    Expand, $a^2 + 2abm_2^2 + b^2 m_2^4 = 4h^2 m_2^2$.

    Divide throughout by $ab$, $\frac{a}{b} + 2m_2^2 + \left(\frac{b}{a}\right)m_2^4 = \left(4\frac{h^2}{ab}\right)m_2^2$.

    Now use $m_2^3 = \frac{a}{b}$.

    Then $m_2^4 = m_2 * m_2^3 = m_2 * \left(\frac{a}{b}\right)$ and $m_2^2 = \frac{\frac{a}{b}}{m_2}$.

    Substitute these into the equation and simplify. After simplification, we obtain $\frac{a+b}{h} + \frac{8h^2}{ab} = 6$.

49. Each represents a pair of straight lines through the origin. Let a common line have slope $m$. Then it must satisfy both equations.

    Putting $y = mx$ in each equation, we get $a + 2hm + bm^2 = 0$ and $a' + 2h'm + b'm^2 = 0$.

    Thus $m$ is a common root of the two quadratic equations $bm^2 + 2hm + a = 0$ and $b'm^2 + 2h'm + a' = 0$.

    For these two quadratics to have a common root, the condition is $\left(ab' - a'b\right)^2 = 4(ah' - a'h)(hb' - h'b)$.

50. Comparing given equation with the general equation in second degree gives us

    $a = 3, 2h = -5$ and $b = 4$.

    We know that the equation of bisector of angles is given by $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

    Substituting the values yields $5x^2 - 2xy - 5y^2 = 0$.

51. The line $x + 2y = 7$ will be equally inclined to the lines $135x^2 - 136xy + 33y^2 = 0$ if it is parallel to bisector of lines of $135x^2 - 136xy + 33y^2 = 0$.

    Comparing given equation with the general equation in second degree gives us

$a = 135, 2h = -136$ and $b = 33$.

Equation of bisectors is given by $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

Substituting the values we obtain $2x^2 + 3xy - 2y^2 = 0 \Rightarrow (x+2y)(2x-y) = 0$.

Clearly, the given line is parallel to $x + 2y = 0$.

52. The two pair of straight lines will be equally inclined if they have the same bisectors.

For $ax^2 + 2hxy + by^2 = 0$ the bisectors are given by $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

For $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ the bisectors are given by $\frac{x^2-y^2}{a^2-b^2} = \frac{xy}{h(a+b)} \Rightarrow \frac{x^2-y^2}{a-b} = \frac{xy}{h}$.

Thus, bisectors are same for them, and hence, they are equally inclined.

53. The bisectors of $ax^2 + 2hxy + by^2 = 0$ are given by $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

$\Rightarrow hx^2 - (a-b)xy - hy^2 = 0$, whose bisectors are given by

$\frac{x^2-y^2}{2h} = \frac{xy}{-\frac{a-b}{2}} \Rightarrow (a-b)(x^2-y^2) + 4hxy = 0$.

54. Equation of the bisectors for the second equation is $\frac{x^2-y^2}{1-(-1)} = \frac{xy}{-}q \Rightarrow qx^2 + 2xy - qy^2 = 0$, which is given to be the same as first equation.

Comparing coefficients $\frac{1}{q} = \frac{-2p}{2} = \frac{1}{q} \Rightarrow pq = -1$.

55. Given pair of lines is $ax^2 + 2hxy + by^2 = \lambda(x^2+y^2)$

The equation of bisectors is given by $\frac{x^2-y^2}{a-\lambda-b+\lambda} = \frac{xy}{h}$, which is independent of $\lambda$. Hence, proven.

56. Equation of bisectors for lines $ax^2 + acxy + cy^2 = 0$ is given by

$\frac{x^2-y^2}{a-c} = \frac{xy}{\frac{ac}{2}} = \frac{2xy}{ac}$.

Equation of bisectors for lines $\left(3+\frac{1}{c}\right)x^2 + xy + \left(3+\frac{1}{a}\right)y^2 = 0$ us given by

$\frac{x^2 0 y^2}{3+\frac{1}{c}-3-\frac{1}{a}} = \frac{xy}{\frac{1}{2}} \Rightarrow \frac{x^2-y^2}{a-c} = \frac{xy}{\frac{ac}{2}} = \frac{2xy}{ac}$, which is same as previous bisectors.

57. The lines will be equally inclined if their bisectors are parallel. The bisectors of $2x^2 + 6xy + y^2 = 0$ are given by

$\frac{x^2-y^2}{2-1} = x\frac{y}{3} \Rightarrow 3(x^2-y^2) = xy$.

The bisectors of $4x^2 + 18xy + y^2 = 0$ are given by

$\frac{x^2-y^2}{4-1} = \frac{xy}{9} \Rightarrow 3(x^2-y^2) = xy$.

Since the bisectors are same the lines are equally inclined.

58. Since the rotation of lines balance each other the bisectors in new position will be same as bisectors of the original position.

Thus, equation of bisectors of angles between the lines $x^2 - 2pxy - y^2 = 0$ is given by

$\frac{x^2-y^2}{2} = -\frac{xy}{p}$.

59. Let the pair of lines be given by the homogeneous second degree equation $ax^2 + 2hxy + by^2 = 0$.

Since one of these lines is the bisector of the angle between the coordinate axes, it must be either $y = x$ or $y = -x$, because these are the angle bisectors of the axes.

Substitute $y = x$ into the given equation $ax^2 + 2hxx + bx^2 = 0$ which simplifies to $(a + 2h + b)x^2 = 0$.

For this to represent a line, the coefficient must be zero, so $a + b + 2h = 0$.

Now substitute $y = -x$ into the equation $ax^2 + 2hx(-x) + bx^2 = 0$

which simplifies to $(a - 2h + b)x^2 = 0$. Thus we get $a + b - 2h = 0$.

In either case, the condition for one of the lines to be an angle bisector is that either $a + b + 2h = 0$ or $a + b - 2h = 0$.

Both can be written together as $a + b = \pm 2h$.

Squaring both sides gives $(a + b)^2 = 4h^2$.

60. First consider the pair of lines $x^2 + xy - 2y^2 + 4x - y + 3 = 0$.

Assume a line through the origin has slope $m$, so it is $y = mx$. Substituting in the homogeneous equation gives $x^2 + mx^2 - 2m^2x^2 = 0$.(we consider only homogeneous part)

After dividing by $x^2$ we obtain $1 + m - 2m^2 = 0$.

Thus the slopes of the two given lines satisfy $2m^2 - m - 1 = 0$.

Let the roots be $m_1$ and $m_2$. Then $m_1 + m_2 = \frac{1}{2}$ and $m_1 m_2 = -\frac{1}{2}$.

The slopes of the angle bisectors satisfy $(1 - m_1 m_2)m^2 + (m_1 + m_2)m - (1 - m_1 m_2) = 0$.

Substituting the values gives $\left(1 + \frac{1}{2}\right)m^2 + \frac{1}{2}m - \left(1 + \frac{1}{2}\right) = 0$.

Multiplying by 2 simplifies this to $3m^2 + m - 3 = 0$.

Hence the bisectors through the origin have slopes satisfying $3m^2 + m - 3 = 0$.

Now the required lines pass through $(1, 2)$ and are parallel to these bisectors, so their form is $y - 2 = m(x - 1)$.

Eliminating $m$ using the quadratic condition gives the combined equation $3(y - 2)^2 + (y - 2)(x - 1) - 3(x - 1)^2 = 0$.

61. Given line is $fx - gy = \lambda \Rightarrow \frac{fx - gy}{\lambda} = 1$.

Given equation is $x^2 + hxy - y^2 + gx + fy = 0$

$\Rightarrow x^2 + hxy - y^2 + gx\frac{fx-gy}{\lambda} + fy\frac{fx-gy}{\lambda} = 0$

$\Rightarrow (\lambda + fg)x^2 + (\lambda h - g^2 + f^2)xy - (\lambda + fg)y^2 = 0$

The above equation represents two lines through origin to the point of intersection of given line and curve.

Coeff. of $x^2$ + Coeff. of $y^2 = 0$. Hence, the lines are perpendicular to each other.

62. $y-3x=2 \Rightarrow \frac{y-3x}{2}=1$. Now equation of the curve is $x^2+2xy+3y^2+4x+8y-11=0$

$\Rightarrow x^2+2xy+3y^2+4\frac{x(y-3x)}{2}+8\frac{y(y-3x)}{2}-11\frac{(y-3x)^2}{4}=0$

$\Rightarrow 7x^2-2xy-y^2=0$.

This is the equation of the lines passing through origin and point of intersection of the given line and curve.

Angle between lines is $\theta=\tan^{-1}\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|=\tan^{-1}\frac{2\sqrt{2}}{5}$.

63. We can write $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(-\frac{lx+my}{n}\right)^2$

$\Rightarrow x^2\left(\frac{1}{a^2}-\frac{l^2}{n^2}\right)-\frac{2ml}{n^2}xy+y^2\left(\frac{1}{b^2}-\frac{m^2}{n^2}\right)=0$

which is of the form $Ax^2+2Hxy+By^2=0$ i.e. the lines pass through origin and point of intersection of the given curve and the given line.

The lines obtained will coincident if $H^2-AB=0$. Substituting the values yields $a^2l^2+b^2m^2=n^2$. Hence, proved.

64. The pair of lines which joing the origin to the point of intersection of the two gives curves can be obtained by making the given curves homogeneous.

Multiplying first with $g_1$ and second with $g$ and subtracting yields

$(ag_1-a_1g)x^2+2(hg_1-h_1g)xy+(bg_1-b_1g)y^2=0$

These lines will be perpendicular to each other if $ag_1-a_1g+bg_1-b_1g=0$

$\Rightarrow g(a_1+b_1)=g_1(a+b)$.

65. The required lines pass through the origin and the points of intersection of $3x+4y=5$ and $2x^2+3y^2=5$.

Let the slope of such a line be $m$, so its equation is $y=mx$.

Substitute $y=mx$ into the line $3x+4(mx)=5$ which gives $x(3+4m)=5$ and hence $x=\frac{5}{3+4m}$.

Now substitute $x$ and $y=mx$ into the curve $2x^2+3y^2=5$.

This gives $2x^2+3m^2x^2=5$ or $x^2(2+3m^2)=5$.

Substitute $x=\frac{5}{3+4m}$, $\left(\frac{5}{3+4m}\right)^2(2+3m^2)=5$.

This simplifies to $m^2+24m-1=0$.

Thus the slopes satisfy $m^2+24m-1=0$.

Replacing $m$ by $\frac{y}{x}$ gives the combined equation of the pair of lines $\left(\frac{y}{x}\right)^2+24\left(\frac{y}{x}\right)-1=0$.

Multiplying by $x^2$ we obtain $y^2+24xy-x^2=0$.

Hence the required pair of straight lines is $y^2+24xy-x^2=0$.

66. Let the required lines through the origin have slope $m$, so $y=mx$.

From $3x-2y=1$, substitute $y=mx$, $x(3-2m)=1$ so $x=\frac{1}{3-2m}$.

Substitute in $3x^2+5xy-3y^2+2x+3y=0 \Rightarrow x^2(3+5m-3m^2)+x(2+3m)=0$.

Put $x=\frac{1}{3-2m}$ and simplify $3+5m-3m^2+(2+3m)(3-2m)=0$.

This gives $9m^2-10m-9=0$.

If slopes are $m_1$ and $m_2$, then $m_1m_2=-1$. Hence, the lines are perpendicular.

67. Given $y=mx+c$ and $x^2+y^2=a^2$.

Substituting $x^2+(mx+c)^2=a^2 \Rightarrow (1+m^2)x^2+2mcx+(c^2-a^2)=0$

Let roots be $x_1, x_2$. Then $x_1+x_2=-\frac{2mc}{1+m^2}$, and $x_1x_2=\frac{c^2-a^2}{1+m^2}$

Slopes of lines from origin are $s_1=m+\frac{c}{x_1}$, $s_2=m+\frac{c}{x_2}$

Product is $s_1s_2=m^2+mc\left(\frac{1}{x_1}+\frac{1}{x_2}\right)+\frac{c^2}{x_1x_2}$

Using identity: $\frac{1}{x_1}+\frac{1}{x_2}=\frac{x_1+x_2}{x_1x_2}$

Simplifying, $s_1s_2=\frac{c^2-m^2a^2}{c^2-a^2}$

For perpendicular lines: $s_1s_2=-1$

So, $\frac{c^2-m^2a^2}{c^2-a^2}=-1 \Rightarrow 2c^2=a^2(1+m^2)$.

68. Let the required lines from origin be $y=kx$.

Point of intersection satisfies both: $lx+mkx+n=0 \Rightarrow x(l+mk)+n=0 \Rightarrow x=-\frac{n}{l+mk}$

Also from parabola: $(kx)^2=4ax \Rightarrow k^2x^2=4ax$

Substituting $x$, $k^2\left(-\frac{n}{l+mk}\right)^2=4a\left(-\frac{n}{l+mk}\right)$

Simplifying, $k^2n^2=-4an(l+mk)$

Rearrangeing $nk^2+4amk+4al=0$

So slopes $k_1, k_2$ satisfy: $k_1k_2=4a\frac{l}{n}$

Hence the required lines are given by: $y=kx$ where $k$ satisfies $nk^2+4amk+4al=0$

Condition for perpendicularity is $k_1k_2=-1$

So, $4a\frac{l}{n}=-1 \Rightarrow 4al+n=0$.

69. Expanding $(x-3)^2+(y-4)^2=c^2$ gives $x^2+y^2-6x-8y+(25-c^2)=0$.

Homogenizing with $\frac{4x+3y}{24}=1$ yields the joint equation of lines from the origin.

$-1200xy+(25-c^2)(16x^2+24xy+9y^2)=0$

For perpendicular lines the condition is $a+b=0$, where $a$ and $b$ are the coefficients of $x^2$ and $y^2$:

$16(25-c^2)+9(25-c^2)=0 \Rightarrow 25(25-c^2)=0 \Rightarrow c=\pm 5.$

70. Homogenizing $x^2+y^2=1$ with $y-mx=1$ gives $x^2+y^2-(y-mx)^2=0$, which expands to $x^2+y^2-y^2+2mxy-m^2x^2=0$, i.e. $(1-m^2)x^2+2mxy=0$.

    For perpendicular lines, $a+b=0$, so the sum of coefficients of $x^2$ and $y^2$ must vanish, i.e. $(1-m^2)+0=0$, giving $m^2=1$, hence $m=\pm 1$.

71. Rewrite the circle as $x^2+y^2-2hx-2ky+(h^2+k^2-c^2)=0$ and homogenize using $1=\frac{kx+hy}{2hk}$:

    $x^2+y^2-2hx\cdot\frac{kx+hy}{2hk}-2ky\cdot\frac{kx+hy}{2hk}+(h^2+k^2-c^2)\cdot\left(\frac{kx+hy}{2hk}\right)^2=0.$

    The coefficient of $x^2$ is $1-1+\frac{h^2+k^2-c^2}{4h^2}=(h^2+k^2-c^2)\frac{a}{4h^2}$ and of $y^2$ is $\frac{h^2+k^2-c^2}{4k^2}$. Applying the perpendicularity condition $a+b=0$:

    $(h^2+k^2-c^2)\left[\left(\frac{1}{4h^2}\right)+\left(\frac{1}{4k^2}\right)\right]=0 \Rightarrow (h^2+k^2-c^2)\cdot\frac{h^2+k^2}{4h^2k^2}=0.$

    Since $h,k\neq 0$, the second factor is nonzero, so $h^2+k^2=c^2$.

72. Let the straight lines represented be $y=m_1x$ and $y=m_2x$ for $ax^2+2hxy+by^2=0$

    Then, $m_1+m_2=-\frac{2h}{b}$ and $m_1m_2=\frac{a}{b}$

    The lines perpendicular to these would be $x+m_1y=0$ and $x+m_2y=0$. Thus, combined equation is

    $(x+m_1y)(x+m_2y) \Rightarrow bx^2-2hxy+ay^2=0.$

73. Let one of the lines have slope $m$ then another line will have slope $m\lambda$. Thus,

    $(y-mx)(y-m\lambda x)=ax^2+2bxy+by^2 \Rightarrow m(1+\lambda)=-\frac{2h}{b}$ and $\lambda m^2=\frac{a}{b}$

    $\Rightarrow \frac{(1+\lambda)^2}{\lambda}=\frac{4\frac{h^2}{b^2}}{\frac{a}{b}}=\frac{4h^2}{ab}.$

74. Let $(y-m_1x)(y-m_2x)=ax^2+2bxy+by^2 \Rightarrow m_1+m_2=-\frac{2h}{b}, m_1m_2=\frac{a}{b}$

    Let $p_1$ and $p_2$ be the length of the perpendiculars from $(x_1,y_1)$ on the two lines. Then

    $p_1p_2=\frac{y_1-m_1x_1}{\sqrt{1+m_1^2}}\frac{y_1-m_2x_1}{\sqrt{1+m_2^2}}=\frac{y_1^2-(m_1+m_2)x_1y_1+m_1m_2x_1^2}{\sqrt{1+m_1^2+m_2^2+m_1^2m_2^2}}$

    $=\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}.$

75. Let $(y-m_1x)(y-m_2x)=ax^2+2bxy+by^2 \Rightarrow m_1+m_2=-\frac{2h}{b}, m_1m_2=\frac{a}{b}.$

    If $m_1=\tan\theta_1$ and $m_2=\tan\theta_2$ then the equation of lines making an angle of $45^\circ$ with these lines will be given by

    $(y-\tan(\theta_1-45^\circ))(y-\tan(\theta_2-45^\circ))=0$

    $\left(y-\frac{m_1-1}{1+m_1}x\right)\left(y-\frac{m_2-1}{1+m_2}x\right)=[(1+m_1)y-(m_1-1)x][(1+m_2)y-(m_2-1)x]=0$

$(1+m_1+m_2+m_1m_2)y^2+[m_1m_2-(m_1+m_2)+1]x^2+2(m_1m_2-1)xy=0$

Substituting the values yields $(a+2b+b)x^2-2(a-b)xy+(a-2h+b)y^2=0$.

76. Let the equation of any line through the origin $(0,0)$ be $y-mx=0$.

Distance from $(\alpha,\beta)$ to the line is

$\left|\frac{\beta-m\alpha}{\sqrt{1+m^2}}\right|=d\Rightarrow\frac{(\beta-m\alpha)^2}{1+m^2}=d^2$

Putting $m=\frac{y}{x}$ in the above equation yields

$(\alpha y-\beta x)^2=d^2(x^2+y^2)$.

77. Let the slope of the lines given by first equation be $m_1$ and $m_2$. Then

$m_1+m_2=-\frac{2h}{b}$ ... (1) and $m_1m_2=\frac{a}{b}$ ... (2)

Then the slope of the lines of the second equation will be $m_1$ and $-\frac{1}{m_2}$ and

$m_1-\frac{1}{m_2}=-\frac{2h_1}{b_1}$ ... (3) and $-\frac{m_1}{m_2}=\frac{a_1}{b_2}$ ... (4)

Multiplying (2) and (4) gives

$m_1^2=-\frac{a_1a}{b_1b}\Rightarrow m_1=\sqrt{-\frac{a_1a}{b_1b}}$. From (4), $m_2=-\frac{b_1}{a_1}m_1=-\frac{b_1}{a_1}\sqrt{-\frac{a_1a}{b_1b}}$

Substituting the values of $m_1$ and $m_2$ in (1) yields

$b\sqrt{-\frac{a_1a}{b_1b}}=\frac{2a_1h}{b_1-a_1}\Rightarrow\frac{1}{2}\sqrt{-a_1ab_1b}=\frac{a_1b_1h}{b_1-a_1}$

Substituting $m_1$ and $m_2$ in (3) yields

$\frac{hab}{b-a}=\frac{1}{2}\sqrt{-aa_1bb_1}$. Hence proven.

78. Let the equations of parallel lines represented by the given equation are $lx+my+n=0$ and $lx+my+n_1=0$, then

$(lx+my+n)(lx+my+n_1)=ax^2+2hxy+by^2+2gx+2fy+c$

Comparing coefficients yields

$l^2=a, m^2=b, nn_1=c, m(n+n_1)=2f, l(n+n_1)=2g, 2lm=2h$

$\Rightarrow h=lm\Rightarrow h^2=l^2m^2=ab\Rightarrow\frac{a}{h}=\frac{h}{b}$

$\frac{2g}{2f}=\frac{l}{m}=\frac{lm}{m^2}=\frac{h}{b}$. Hence, $\frac{a}{h}=\frac{h}{b}=\frac{g}{f}$.

The distance is given by $\frac{|n-n_1|}{\sqrt{l^2-m^2}}$

Now $(n-n_1)^2=(n+n_1)^2-2n_1n=4\left(\frac{g^2-ca}{a}\right)\Rightarrow|n-n_1|=2\sqrt{\frac{g^2-ca}{a}}$ and $l^2+m^2=a+b$

$d=2\sqrt{\frac{g^2-ca}{a(a+b)}}$.

79. Let two straight lines $lx+my+n=0$ and $l_1x+m_1y+n_1=0$, then $(lx+my+n)(l_1x+m_1y+n_1)=(ax^2+2hxy+by^2+2gx+2fy+c)$

Comparing coefficients yields, $ll_1 = a, mm_1 = b, nn_1 = c, lm_1 + l_1m = 2h, ln_1 + l_1n = 2g, mn_1 + m_1n = 2f$

Solving the two straight line equations we get point of intersection as $\left(\frac{mn_1-m_1n}{lm_1-l_1m}, \frac{nl_1-n_1l}{lm_1-l_1m}\right)$

Distance squared from origin is $\left(\frac{mn_1-m_1n}{lm_1-l_1m}\right)^2 + \left(\frac{nl_1-n_1l}{lm_1-l_1m}\right)^2$

Substituting values yields $\frac{c(a+b)-f^2-g^2}{ab-h^2}$

80. Let the lines be $lx + my + n = 0$ and $l'x + m'x + n' = 0$ represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. Then

    $(lx + my + n)(l'x + m'y + n') = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ which implies

    $ll' = a, mm' = b, nn' = c, lm' + l'm = 2h, ln' + l'n = 2g, mn' + m'n = 2f$

    Since the lines are equidistant, therefore $\frac{|n|}{\sqrt{l^2+m^2}} = \frac{|n'|}{\sqrt{l'^2+m'^2}}$

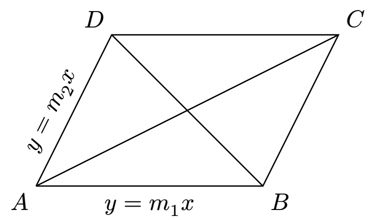
    $\Rightarrow n^2l'^2 + n^2m'^2 = n'^2;^2 + n'^2m^2 \Rightarrow (nl' - n'l)(nl' + n'l) = (mn' - m'n)(mn' + m'n)$

    Squaring yields

    $\left[(nl' + n'l)^2 - 4ll'nn'\right](nl' + n'l)^2 = \left[(mn' + m'n)^2 - 4mm'nn'\right](mn' + m'n)^2$

    $\Rightarrow f^4 - g^4 = c(bf^2 - ag^2)$ upon substitution and simplification.

81. Let the lines $y = m_1x$ and $y = m_2x$ be two parallel lines intersecting at the origin $A$. Then $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$




    Clearly, the diagonal $lx + my = 1$ is $BD$ as it does not pass through origin. Solving this digonal with the two parallel lines we get coordinates of $B = \left(\frac{1}{l+mm_1}, \frac{m}{1+mm_1}\right)$ and $D = \left(\frac{1}{l+mm_2}, \frac{m}{1+mm_2}\right)$.

    Let $H$ be the point of intersection of the two diagonals then $H = \left(\frac{bl-hm}{b(l+mm_1)(l+mm_2)}, \frac{am-hl}{b(l+mm_1)(l+mm_2)}\right)$ and let $A$. be the origin.

    Equation of other diagonal passing through the origin and $H$ is given by $y(bl - hm) = x(am - hl)$.

82. Let $\triangle ABC$ be the triangle such that $AB, AC$ and $BC$ are given by $y = m_1x, y = m_2x$ and $lx + my = 1$ such that

$(y - m_1x)(y - m_2x) = ax^2 + 2hxy + by^2$. Let $AL$ and $BM$ be perpendiculars from $A$ and $B$ on opposite sides.

Equation of $AL$ which is perpendicular to $BC$ and passes through $A$, the origin, is given by

$mx - ly = 0 \Rightarrow \frac{x}{l} = \frac{y}{m} = k(\text{say})$

Orthocenter will be on this line and its coordinates are $(kl, km)$ for such suitable values of $k$ for which the point may also lie on $BM$, perpendicular from $B$ on $AC$, which is perpendicular to line $y = m_2x$.

Thus, we find $H$, the orthocenter as $H = \left(\frac{1}{l+mm_1}, \frac{m_1}{l+mm_1}\right)$

So equation of $BM$ is $m_2y + x = \frac{1+m_1m_2}{l+mm_1}$

Substituting $(kl, km)$ yields $k = \frac{1+m_1m_2}{(l+mm_1)(l+mm_2)} = \frac{a+b}{am^2-2hlm+bl^2}$

Thus, $\frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2-2hlm+bl^2}$.

83. Let $y = m_1x$ and $y = m_2x$ be the lines represented by $ax^2 + 2hxy + by^2 = 0$ i.e. $(y - m_1x)(y - m_2x) = ax^2 + 2hxy + by^2$ then

$m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$. Let these lines meet $lx + my + n = 0$ at $A$ and $B$ respectively, then $C$ will be the origin.

Solving the equations we get $A = \left(-\frac{n}{l+mm_1}, -\frac{m_1n}{l+mm_1}\right)$ and $B = \left(-\frac{n}{l+mm_2}, -\frac{m_2n}{l+mm_2}\right)$

Then $\Delta = \frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}$.

84. Let $\theta$ be the acute angle between the lines of $(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$

then $\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right| = \sqrt{3}$

Writing $(A^2 - 3B^2)x^2 + 8ABxy + (B^2 - 3A^2)y^2 = 0$ as a quadratic equation in $\frac{y}{x}$ we have

$\frac{y}{x} = \left(\frac{-4AB\pm\sqrt{3}(A^2+B^2)}{B^2-3A^2}\right)x$

Taking the positive sign we find angle between these lines and $Ax + By + C = 0$ is also $60^\circ$, and hence, the triangle is equilateral.

Computing altitude from $(0, 0)$ to $Ax + By + C = 0$ we find the area as $\frac{C^2}{\sqrt{3}(A^2+B^2)}$.

85. Let the pair of lines be $(lx + my)^2 - 3(mx - ly)^2 = 0$

This represents two lines through the origin. Rewrite it as $(lx + my)^2 = 3(mx - ly)^2$

Taking square roots $lx + my = \sqrt{3}(mx - ly)$and $lx + my = -\sqrt{3}(mx - ly)$

So the two lines are $lx + my - \sqrt{3}(mx - ly) = 0$ and $lx + my + \sqrt{3}(mx - ly) = 0$

First line $\left(l-\sqrt{3}m\right)x+\left(m+\sqrt{3}l\right)y=0$

Second line $\left(l+\sqrt{3}m\right)x+\left(m-\sqrt{3}l\right)y=0$

Thus $\tan\theta=\left|\frac{-2\sqrt{3}(l^2+m^2)}{-2(l^2+m^2)}\right|=\sqrt{3}$

So $\theta=60^\circ$. Hence the two lines are inclined at $60^\circ$

Now consider the third line $lx+my+n=0$

The first two lines pass through origin and make angle $60^\circ$. The third line does not pass through origin so it intersects both lines forming a triangle.

Since the angle between the two lines is$60^\circ$ and the third line cuts them symmetrically with equal intercept structure, the triangle formed has all angles$60^\circ$.

Hence the three lines form an equilateral triangle.

86. Given $12x^2+7xy-12y^2=0$

Putting $y=mx$ $12+7m-12m^2=0\Rightarrow 12m^2-7m-12=0$

$m=\frac{7\pm25}{24}\Rightarrow m_1=\frac{4}{3},\ m_2=-\frac{3}{4}$

Lines are $4x-3y=0,\ 3x+4y=0$

Given $12x^2+7xy-12y^2-x+7y-1=0$

$\Rightarrow(4x-3y+k_1)(3x+4y+k_2)=0$

Comparing yields $3k_1+4k_2=-1,\ 4k_1-3k_2=7,\ k_1k_2=-1$

$k_1=1,\ k_2=-1$

Lines are $4x-3y+1=0,\ 3x+4y-1=0$

Product of slopes are $\left(\frac{4}{3}\right)\left(-\frac{3}{4}\right)=-1$

Distances between $4x-3y=0$ and $4x-3y+1=0$ is $\frac{1}{5}$. And between $3x+4y=0$ and $3x+4y-1=0$ is $\frac{1}{5}$

Parallel pairs, perpendicular adjacent sides, equal distances gives a square.

87. $x^2-3xy+2y^2=0\Rightarrow x(x-2y)-y(x-2y)=(x-y)(x-2y)=0$

Lines parallel to these are $x-y+k=0$ and $x-2y+k'=0$. Since both lines pass through $(1,1)$, therefore $k=0,k'=1$

So the combined equation is $(x-y)+(x-2y+1)=x^2-3xy+2y^2+x-y=0$.

88. $2x^2-xy-y^2=2x(x-y)+y(x-y)\Rightarrow(x-y)(2x+y)=0$

The equation from whcih these were reflected will be bisectors which are perpendicular to each other. Equation of bisectors is given by
$\frac{x^2-y^2}{2+1}=\frac{xy}{-}\left(\frac{1}{2}\right)=-2xy\Rightarrow x^2-y^2+6xy=0.$

89. Given $x\cos\alpha+y\sin\alpha=1$ and $x^2+y^2=a^2$

Homogenizing yields $x^2+y^2=a^2(x\cos\alpha+y\sin\alpha)^2$

$\Rightarrow x^2+y^2-a^2(x\cos\alpha+y\sin\alpha)^2=0$

Expanding $(1-a^2\cos^2\alpha)x^2 - 2a^2\sin\alpha\cos\alpha xy + (1-a^2\sin^2\alpha)y^2 = 0$

For perpendicular pair, $(1-a^2\cos^2\alpha) + (1-a^2\sin^2\alpha) = 0$

$\Rightarrow a^2 = 2 \Rightarrow a = \pm\sqrt{2}$.

90. Given $ax^2 + 2hxy + by^2 = 0$

For rotation of axes by an angle $\theta$ (anticlockwise), the standard transformation is $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$

Now taking $\theta = 90^{\circ}$

We know $\cos 90^{\circ} = 0$, $\sin 90^{\circ} = 1$

Substituting, $x = x'(0) - y'(1) = -y'$ and $y = x'(1) + y'(0) = x'$

Hence, for $90^{\circ}$ rotation: $x = -y'$, $y = x'$

Substituting $a(-y')^2 + 2h(-y')(x') + b(x')^2 = 0$

$\Rightarrow ay'^2 - 2hx'y' + bx'^2 = 0$

Rewriting $bx'^2 - 2hx'y' + ay'^2 = 0$, thus equation is $bx^2 - 2hxy + ay^2 = 0$.

91. Let the equations make angles of $\alpha$ and $\beta$ with the positive direction of $x$-axis. Then $(y - \tan\alpha x)(y - \tan\beta x) = ax^2 + 2hxy + by^2 \Rightarrow \tan\alpha + \tan\beta = -\frac{2h}{b}, \tan\alpha\tan\beta = \frac{a}{b}$

$y = 0$ is $x$-axis. Since the lines make angle $\alpha, \beta$ after reflection they will make same angle with negtive direction of $x$-axis. Thus, reflected lines will make angle $\pi - \alpha$ with positive direction of $x$-axis.

Thus, new equation is $(y + \tan\alpha)(y + \tan\beta) = ax^2 - 2hxy + by^2 = 0$.

92. Given the pair of straight lines $ax^2 + 2hxy + by^2 = 0$

Let the lines be $y = m_1 x$ and $y = m_2 x$ where $bm^2 + 2hm + a = 0$

Hence $m_1 + m_2 = -2\frac{h}{b}$ and $m_1 m_2 = \frac{a}{b}$

The perpendicular distance from $(x', y')$ to the line $y = mx$ is $\frac{|mx' - y'|}{\sqrt{m^2+1}}$

The sum of the squares of the perpendiculars is $S = \frac{(m_1 x' - y')^2}{m_1^2+1} + \frac{(m_2 x' - y')^2}{m_2^2+1}$

On combining the two terms into a single fraction and expanding, and then using the relations $m_1 + m_2 = -2\frac{h}{b}$ and $m_1 m_2 = \frac{a}{b}$, the numerator simplifies to $4h^2(x'^2 + y'^2) + 4h(a+b)x'y' + 2(a-b)(ax'^2 - by'^2)$

The denominator simplifies to $(a-b)^2 + 4h^2$

Therefore the sum of the squares of the perpendiculars is $\frac{[4h^2(x'^2+y'^2)+4h(a+b)x'y'+2(a-b)(ax'^2-by'^2)]}{[(a-b)^2+4h^2]}$.

93. Let the pair of lines $ax^2 + 2hxy + by^2 = 0$ represent two lines through the origin. Let their slopes be $m_1$ and $m_2$ so that $bm^2 + 2hm + a = 0$. Hence $m_1 + m_2 = -2\frac{h}{b}$ and $m_1 m_2 = \frac{a}{b}$.

The triangle is formed by these two lines and $lx + my + n = 0$.

Intersection with $y = m_1 x$ gives $x_1 = -\frac{n}{l+mm_1}$ and $y_1 = -m_1 \frac{n}{l+mm_1}$.

Intersection with $y = m_2 x$ gives $x_2 = -\frac{n}{l+mm_2}$ and $y_2 = -m_2 \frac{n}{l+mm_2}$.

The centroid is $x = \frac{x_1+x_2+x_3}{3}$ and $y = \frac{y_1+y_2+y_3}{3}$.

So $x = -\frac{n}{3}\left(\frac{1}{l+mm_1} + \frac{1}{l+mm_2}\right)$ and $y = -\frac{n}{3}\left(\frac{m_1}{l+mm_1} + \frac{m_2}{l+mm_2}\right)$.

Now $\frac{1}{l+mm_1} + \frac{1}{l+mm_2} = \frac{2l+m(m_1+m_2)}{(l+mm_1)(l+mm_2)}$

and $(l + mm_1)(l + mm_2) = l^2 + lm(m_1 + m_2) + m^2 m_1 m_2$.

Substitute $m_1 + m_2 = -2\frac{h}{b}$ and $m_1 m_2 = \frac{a}{b}$ to get $(l + mm_1)(l + mm_2) = \frac{bl^2-2hlm+am^2}{b}$.

Also $2l + m(m_1 + m_2) = 2l - 2h\frac{m}{b}$.

Thus $x = -\frac{n}{3}\left(\frac{2l-2h\frac{m}{b}}{\frac{bl^2-2hlm+am^2}{b}}\right) = -2n\frac{bl-hm}{3(bl^2-2hlm+am^2)}$.

Similarly $\frac{m_1}{l+mm_1} + \frac{m_2}{l+mm_2} = \frac{l(m_1+m_2)+2mm_1m_2}{(l+mm_1)(l+mm_2)}$

$= \frac{-2h\frac{l}{b}+2a\frac{m}{b}}{\frac{bl^2-2hlm+am^2}{b}} = \frac{-2hl+2am}{bl^2-2hlm+am^2}$.

Hence $y = -\frac{n}{3}\left(\frac{-2hl+2am}{bl^2-2hlm+am^2}\right) = -2n\frac{am-hl}{3(bl^2-2hlm+am^2)}$.

Therefore the centroid is $\left(\frac{-2n(bl-hm)}{3(bl^2-2hlm+am^2)}, \frac{-2n(am-hl)}{3(bl^2-2hlm+am^2)}\right)$.

94. Let the two sides of the triangle through the origin be represented by $ax^2 + 2hxy + by^2 = 0$

Let their slopes be $m_1$ and $m_2$. Then they satisfy $bm^2 + 2hm + a = 0$

Hence $m_1 + m_2 = -2\frac{h}{b}$ and $m_1 m_2 = \frac{a}{b}$. The two sides are therefore $y = m_1 x$ and $y = m_2 x$.

Let the third side be $lx + my + n = 0$ and let the orthocenter be $(l, m)$.

The altitude from $(l, m)$ to the line $y = m_1 x$ is perpendicular to it, so its slope is $-\frac{1}{m_1}$. Hence its equation is $y - m = -\frac{1}{m_1}(x - l)$.

Since this altitude lies along the other side $y = m_2 x$, substitute $y = m_2 x$ to get $m_2 x - m = -\frac{1}{m_1}(x - l)$.

On simplifying this relation and using $m_1 m_2 = \frac{a}{b}$, a relation between $l$ and $m$ is obtained. A similar relation is obtained from the other altitude. Combining these and eliminating the parameters gives the equation of the third side.

After simplification using $m_1 + m_2 = -2\frac{h}{b}$ and $m_1 m_2 = \frac{a}{b}$, the equation of the third side is found to be $(a + b)(lx + my) = am^2 - 2hlm + bl^2$.

95. Given $x^2 + 4xy + y^2 = 0$. Putting $y = mx$, $m^2 + 4m + 1 = 0$.

$m_1 = -2 + \sqrt{3}$ and $m_2 = -2 - \sqrt{3}$.

Angle between the lines $\tan\theta = |\frac{m_1 - m_2}{1 + m_1 m_2}| = \sqrt{3}$ so $\theta = 60^\circ$.

Hence, the angle at the origin is $60^\circ$.

The third side is $x - y = 4$.

Distance from origin to this line $= \frac{|0-0-4|}{\sqrt{1^2+(-1)^2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.

This is the altitude of the equilateral triangle.

Let side be $s$. Altitude $= s\frac{\sqrt{3}}{2} = 2\sqrt{2}$.

So $s = 4\frac{\sqrt{2}}{\sqrt{3}}$.

Area $= \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left(16 * \frac{2}{3}\right) = \frac{8}{\sqrt{3}}$.

96. Given the pair of lines $x^2 + 4xy + y^2 = 0$

Put $y = mx$, $m^2 + 4m + 1 = 0 \Rightarrow m_1 = -2 + \sqrt{3}$, $m_2 = -2 - \sqrt{3}$

Angle between the two lines is $\tan\theta = |\frac{m_1 - m_2}{1 + m_1 m_2}| = \sqrt{3}$ so $\theta = 60^\circ$

Now the second line is $x + y + 4\sqrt{6} = 0$. Its slope is $-1$

Check angle with $y = mx$, $\tan\theta_1 = |\frac{m_1 + 1}{1 - m_1}| = \sqrt{3}$ so $\theta_1 = 60^\circ$

$\tan\theta_2 = |\frac{m_2 + 1}{1 - m_2}| = \sqrt{3}$ so $\theta_2 = 60^\circ$

Therefore the triangle is equilateral and the angles are $60^\circ$ each.

97. Given $ax^2 + 2hxy + by^2 = 0$ and $x\cos\alpha + y\sin\alpha = p$

Let slopes of the pair be $m_1, m_2$ where $bm^2 + 2hm + a = 0$

So $m_1 + m_2 = -\frac{2h}{b}$, $m_1 m_2 = \frac{a}{b}$

Points of intersection with $y = mx$ give $x = \frac{p}{\cos\alpha + m\sin\alpha}$, $y = m\frac{p}{\cos\alpha + m\sin\alpha}$

Area of triangle is $\frac{1}{2}\,|x_1 y_2 - x_2 y_1|$

$= \frac{1}{2}p^2\frac{|m_2 - m_1|}{(\cos\alpha + m_1\sin\alpha)(\cos\alpha + m_2\sin\alpha)}$

$(\cos\alpha + m_1\sin\alpha)(\cos\alpha + m_2\sin\alpha) = \cos^2\alpha + \cos\alpha\sin\alpha(m_1 + m_2) + \sin^2\alpha m_1 m_2$

$= \frac{b\cos^2\alpha - 2h\sin\alpha\cos\alpha + a\sin^2\alpha}{b}$

Also $m_2 - m_1 = 2\frac{\sqrt{h^2 - ab}}{b}$

So area becomes $p^2\frac{\sqrt{h^2 - ab}}{b\cos^2\alpha - 2h\sin\alpha\cos\alpha + a\sin^2\alpha}$.

98. Given $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Set $y = 0$ to get intersection with x-axis, $ax^2 + 2gx + c = 0$

Let roots be $x_1, x_2$

So base lies on x-axis with endpoints $(x_1, 0)$ and $(x_2, 0)$

Base length $|x_1 - x_2| = 2\frac{\sqrt{g^2 - ac}}{|}a|$

The pair of lines has angle factor of $\sqrt{h^2 - ab}$

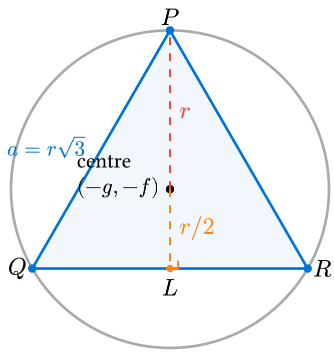
Hence, area of triangle formed by the pair of lines and x-axis is $\frac{g^2-ac}{a\sqrt{h^2-ab}}$.

# 5 Answers of Circles

1. Given circle is $x^2+y^2-\frac{8}{3}x-\frac{10}{3}y+1=0$. We know that if the equation is $x^2+y^2+2gx+2fy+c=0$, then the center is $(-g,-f)$ and radius is $\sqrt{g^2+f^2-c}$. Thus, center of circle is $\left(\frac{4}{3},\frac{5}{3}\right)$ and radius is $\sqrt{\frac{16}{9}+\frac{25}{9}-1}=\frac{4}{3}\sqrt{2}$.

2. Radius of $x^2+y^2=1$ is 1. Radius of $x^2+y^2-2x-6y=6$ is $\sqrt{1^2+3^2+6}=4$. Radius of $x^2+y^2-4x-12y=9$ is $\sqrt{2^2+6^2+9}=7$.

   Thus, radii of the given circles are in A.P. with a common difference of 3.

3. 


   The center will be $C(-g,-f)$ and the radius will be $\sqrt{g^2+f^2-c}$

   From $\triangle\, QLC, QL=CQ\sin 50^{\circ}=\frac{\sqrt{3}}{2}\sqrt{g^2+f^2-c}$

   $QR=2QL=\sqrt{3}\sqrt{g^2+f^2-c}$

   $\Delta=\frac{\sqrt{3}}{4}3.(g^2+f^2-c)=\frac{3\sqrt{3}}{4}(g^2+f^2-c)$.

4. The center of the circle is $(3,-4)$ and radius is $\sqrt{3^2+(-4)^2+25}=7$.

5. Given circle is $x^2+y^2+\frac{4}{5}x-\frac{8}{5}y=\frac{16}{5}$. Cneter is $\left(-\frac{2}{5},\frac{4}{5}\right)$ and radius is $\sqrt{\frac{(-2)^2+4^2+80}{25}}=2$.

6. Center is $(3,1)$ and radius is $\sqrt{3^2+1+6}=4$.

7. Given circle is $x^2+y^2+2x\cos\theta+2y\sin\theta-8=0$. Center is $(-\cos\theta,-\sin\theta)$ and radius is $\sqrt{\cos^2\theta+\sin^2\theta+8}=3$

8. Centers of the given circles are $(0,0),(-3,1)$ and $(6,-2)$. Area of the triangle whose vertices are these three points is given by

   $\Delta=\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ -3 & 1 & 1\\ 6 & -2 & 1\end{vmatrix}=0.$

   Hence, the centers of the given circles are collinear.

9. Radii are $1, \sqrt{1+3^2+6} = 4$ and $\sqrt{2^2+6^2+9} = 7$. Common difference is 3, and hence, the given circles have radii in A.P.
10. Radii are $2, \sqrt{1^2+3^2-\frac{15}{4}} = \frac{5}{2}$ and $\sqrt{2^2+5} = 3$. Common difference is $\frac{1}{2}$, and hence, the given circles have radii in A.P.
11. The line $3x = 4y + 15$ gives $y = \frac{3x-15}{4}$.

    Substituting into the circle equation $x^2 + y^2 = 9 + 4r^2$ gives $x^2 + \left(\frac{3x-15}{4}\right)^2 = 9 + 4r^2$.

    Multiplying through by 16 gives $16x^2 + (3x-15)^2 = 16(9+4r^2)$.

    Expanding gives $16x^2 + 9x^2 - 90x + 225 = 144 + 64r^2$.

    Simplifying gives $25x^2 - 90x + 81 - 64r^2 = 0$.

    Solving for $x$ gives $x = \frac{90 \pm \sqrt{8100 - 100(81-64r^2)}}{50}$.

    This simplifies to $x = \frac{90 \pm 80r}{50} = \frac{9 \pm 8r}{5}$.

    Thus the two points of intersection correspond to $x = \frac{9-8r}{5}$ and $x = \frac{9+8r}{5}$.

    The length of the intercept cut by each circle on the line is $\frac{(9+8r)-(9-8r)}{5} = 16\frac{r}{5}$.

    For successive values of $r$, the difference of intercepts is constant since $\frac{16\times 2}{5} - \frac{16\times 1}{5} = \frac{16}{5}$ and $\frac{16\times 3}{5} - \frac{16\times 2}{5} = \frac{16}{5}$.

    Therefore the circles cut off equal intercepts on the given line.
12. Since point $(4, 6)$ lies on the circle, therefore, radius $= \sqrt{(1-4)^2 + (2-6)^2} = 5$

    Thus, equation of the circle would be $(x-1)^2 + (y-2)^2 = 5^2 \Rightarrow x^2 + y^2 - 2x - 4y - 20 = 0$.
13. Solving the equations of the diameters we obtain the center as $(8, -2)$.

    Thus, equation of the circle is given by $(x-8)^2 + (y+2)^2 = 10^2 \Rightarrow x^2 + y^2 - 16x + 4y - 32 = 0$.
14. Since the line touches the circle, therefore, the perpendicular distance from center will give radius.

    Radius is $\frac{3.5 + 4.12 - 1}{\sqrt{5^2+13^2}} = \frac{62}{13}$.

    Thus, equation of the circle is $(x-3)^2 + (y-4)^2 = \left(\frac{62}{13}\right)^2 \Rightarrow x^2 + y^2 - 6x - 8y + \frac{381}{169} = 0$.
15. Solging the two equation we have $x = \frac{c+1}{3c+2}$. Now as $c \to 1, x = \frac{2}{5} \Rightarrow y = -\frac{1}{25}$.

    The circle passes through $(2, 0)$, so the radius is $\sqrt{\left(2 - \frac{2}{5}\right)^2 + \left(\frac{1}{25}\right)^2}$

    Thus, equation of the circle is $\left(x - \frac{2}{5}\right)^2 + \left(y + \frac{1}{25}\right)^2 = \frac{64}{25} + \frac{1}{625} \Rightarrow 25(x^2+y^2) - 20x + 2y - 60 = 0$.
16. Let the center be $(\alpha, \beta)$. Since it lies on the line $y = x - 1 \therefore \beta = \alpha - 1$

Equation of the circle would be $(x-\alpha)^2+(y-\beta)^2=3^2$. Putting the value of $\beta$ from the above equation, we have

$(x-\alpha)^2+(y-\alpha+1)^2=9$

Since it passes through $(7,3)$, therefore, $(7-\alpha)^2+(4-\alpha)^2=9 \Rightarrow \alpha^2-11\alpha+28=0$

$\alpha=4,7 \Rightarrow \beta=3,6$

So the center is $(4,3)$ or $(7,6)$.

Thus equation of circle is $(x-4)^2+(y-3)^2=3^2$ or $(x-7)^2+(y-6)^2=3^2$ i.e. $x^2+y^2-8x-6y+16=0$ or $x^2+y^2-14x-12y+76=0$.

17. Since the circle touches the axes the center will lie on $x=y$ or $y=-x$.

**Case I:** When the center lies on the line $y=x$.

Solving the two equations we get $x=-3, y=-3$

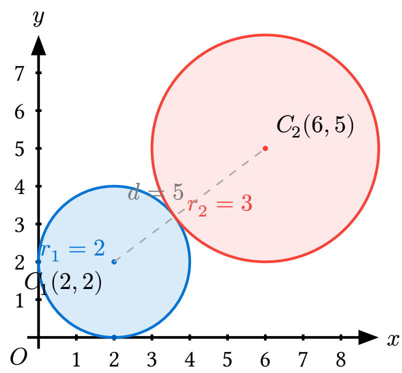
The radius of the circle would be 3. Therefore, the equation of the circle is $(x+3)^2+(y+3)^2=3^2 \Rightarrow x^2+y^2+6x+6y+9=0$.

**Case II:** When the center lies on the line $y=-x$.

Solving the two equations we get $x=1, y=-1$.

The radius of the circle will be 1. Hence, equation of the curcle is $(x-1)^2+(y+1)^2=1 \Rightarrow x^2+y^2-2x+2y+1=0$.

18.




Radius of the first circle is 2 so its center would be $(2,2)$ as it touches both the coordinate axes.

Distance between the centers is $\sqrt{(2-6)^2+(2-5^2)}=5$. Hence, radius of the second circle is $5-2=3$

Hence, equation of the circle is $(x-6)^2+(y-5)^2=3^2 \Rightarrow x^2-12x+36+y^2-10y+25=9$

$\Rightarrow x^2+y^2-12x-10y+52=0$.

19. Half of intercept(chord) is 3. Perpendicualr distance from center to $2x - 5y + 18 = 0$ is

$\frac{2\times3-5\times(-1)+18}{\sqrt{2^2+5^2}} = \sqrt{29}$

Thus, radius of the circle is $\sqrt{\left(\sqrt{29}\right)^2 + 3^2} = \sqrt{38}$.

20. Since the circle touches $y$-axis at $(0, 3)$, therefore, the two circles will be in first two quadrants. Let $C$ and $D$ be the centers of the circles in the first and second quadrant respectively. Let $AB$ be the intercept on $x$-axis i.e. $AB = 8$. Let $CL$ be the perpendicular from $C$ on $x$-axis.

$\therefore AL = 4.\ OP = 3 \therefore CL = 3$

$AC = \sqrt{4^2 + 3^2} = 5 \therefore C = (5, 3)$ and $D = (-5, 3)$

Hence equations of circles are $(x-5)^2 + (y-3)^2 = 5^2$ and $(x+5)^2 + (y-3)^2 = 5^2$ i.e

$x^2 + y^2 - 10x - 6y + 9 = 0$ and $x^2 + y^2 + 10x - 6y + 9 = 0$.

21. $x^2 + 2xy + 3x + 6y = 0 \Rightarrow (x+3)(x+2y) = 0 \Rightarrow x + 3 = 0$ and $x + 2y = 0$

Since these are normals, therefore, their point of intersection will be the center of the circle. Solving gives us $\left(-3, \frac{3}{2}\right)$ as the center of the circle.

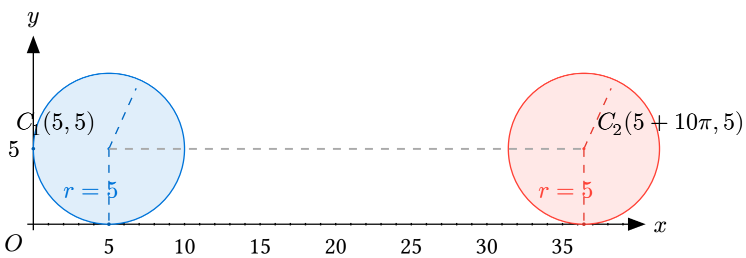
Center of the given circle is $\left(2, \frac{3}{2}\right)$ and its radius is $\sqrt{2^2 + \frac{9}{4}} = \frac{5}{2}$

Distance between centers is 5, which is greater than radius of the given circle.

Since the circle is just large enough to contain the given circle the radius of the required circle is distance between the centers plus radius of the given circle i.e. $5 + \frac{5}{2} = \frac{15}{2}$

Thus, the equation is $(x+3)^2 + \left(y - \frac{3}{2}\right)^2 = \left(\frac{15}{2}\right)^2 \Rightarrow x^2 + y^2 + 6x - 3y - 45 = 0$.

22.




Let $C$ be the center of the circle in original position adn $D$ be the center of its new position. Then $C = (5, 5)$ and $D = (5 + 10\pi, 5)$. Radius is 5.

Thus, equation of the circle in new position is $(x - 5 - 10\pi)^2 + (y-5)^2 = 5^2$

$\Rightarrow x^2 + y^2 - 10(2\pi + 1)x - 10y + 100\pi^2 + 100\pi + 25 = 0$.

23. Equation of tangent to the circle at $\left(2 + \sqrt{3}, 3\right)$ is given by

$\left(2+\sqrt{3}\right)x+3y-2\left(x+2+\sqrt{3}\right)-4(y+3)+16=0 \Rightarrow \sqrt{3}x-y-2\sqrt{3}=0$

This line makes angle of $60^{\circ}$ with $x$-axis. So initial position of the center is $A=(2,4)$ and final position of the center is $B=(2+2\cos 60^{\circ}, 4+2\sin 60^{\circ})=\left(3,4+\sqrt{3}\right)$

Radius of the circle is $\sqrt{2^2+4^2-16}=2$

Thus, equation of the circle in final position is $(x-3)^2+\left(y-4-\sqrt{3}\right)^2=2^2$

$\Rightarrow x^2+y^2-6x-2\left(4+\sqrt{3}\right)y+24+8\sqrt{3}=0.$

24. Let $(\alpha,\beta)$ be the center and $a$ the radius of the required circle, then its equation is

$(x-\alpha)^2+(y-\beta)^2=a^2$. According to the questions $\alpha^2+\beta^2=1$

Since it touches $y=0$ we have $|\beta|=a \Rightarrow \beta=\pm a$

It also touches $y=\sqrt{3}(x+1)$ we have $\frac{|\sqrt{3}\alpha-\beta+\sqrt{3}|}{2}=a \Rightarrow \sqrt{3}\alpha-\beta+\sqrt{3}=\pm 2a$

**Case I:** $\beta=a$ and $\sqrt{3}\alpha-\beta+\sqrt{3}=2a$ gives us $\alpha=\sqrt{3}a-1, \beta=a$

Thus, $\left(\sqrt{3}a-1\right)^2+a^2=1 \Rightarrow a=\frac{\sqrt{3}}{2}$

Hence, center of the circle is $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and radius is $\frac{\sqrt{3}}{2}$.

**Case II:** $\beta=a$ and $\sqrt{3}\alpha-\beta+\sqrt{3}=-2a$ gives us $\alpha=-\frac{a}{\sqrt{3}}-1$

Thus, $\left(\frac{a}{\sqrt{3}}+1\right)^2+a^2=a \Rightarrow 0,-\frac{\sqrt{3}}{2}[a>0]$

**Case III:** $\beta=-a$ and $\sqrt{3}\alpha-\beta+\sqrt{3}=2a$ gives us $\alpha=\frac{a}{\sqrt{3}}-1$

Thus, $\left(\frac{a}{\sqrt{3}}-1\right)+a^2=1 \Rightarrow a=\frac{\sqrt{3}}{2}$

Hence, center of circle is $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ and radius is $\frac{\sqrt{3}}{2}$.

**Case IV:** $\beta=-a$ and $\sqrt{3}\alpha-\beta+\sqrt{3}=-2a$ gives us $\alpha=\sqrt{3}a-1$

Thus, $\left(\sqrt{3}a+1\right)^2+a^2=1 \Rightarrow a=0,-\frac{\sqrt{3}}{2}\ [a>0]$.

25. Let $C(\alpha,\beta)$ be the center of the circle adn $a$ be its radius. According to question

$\frac{|4\alpha-3\beta-24|}{5}=\frac{|4\alpha+3\beta-42|}{5}=\sqrt{(\alpha-2)^2+(\beta-8)}^2$

Solving these gives $\beta=3, \alpha=\frac{33}{4}$ but it is give that $|\alpha|\leq 8$

Putting $\beta=3$ in first and third one we get $\alpha=2,-\frac{182}{9} \Rightarrow \alpha=2$

Hence, equation of the circle is $(x-2)^2+(y-3)^2=(2-2)^2+(3-8)^2=25$.

26. Equation of the circle is $(x-1)^2+(y+5)^2=7^2 \Rightarrow x^2-2x+y^2+10y-23=0$.

27. Equation of the circle is $(x+1)^2+(y+2)^2=\frac{625}{4} \Rightarrow x^2+y^2+2x+4y+-\frac{605}{4}=0$.

28. Given diameters are $2x + y = 6$ and $3x + 2y = 4$. Solving we get $3x + 2(6 - 2x) = 4 \Rightarrow x = 8, y = -10$.

    Thus, equation of the circle is $(x - 8)^2 + (y + 10)^2 = 10^2 \Rightarrow x^2 + y^2 - 16x + 20y + 64 = 0$.

29. Given lines are $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$. Thus, $y = 27 - 4x$ and

    $3x - 2(27 - 4x) - 1 = 0 \Rightarrow x = 5, y = 7$.

    Let the radius of the circle is $r$. Then equation will be $(x - 2)^2 + (y - 3)^2 = r^2$.

    Since it passes theough $(5, 7)$, therefore, $3^2 + 4^2 = r^2 \Rightarrow r = 5$

    Thus, equation will be $x^2 + y^2 - 4x - 6y - 12 = 0$.

30. Given lines are $3x + y = 14$ and $2x + 5y = 18$. From first $y = 14 - 3x$. Putting in the second $2x + 5(14 - 3x) = 18 \Rightarrow x = 4 \Rightarrow y = 2$

    Let the radius of the circle is $r$, then the equation is $(x - 1)^2 + (y - 2)^2 = r^2$

    Since it passes through $(4, 2)$, we have $3^2 + 0 = r^2 \Rightarrow r = 3$.

    Thus, equation of the circle is $x^2 + y^2 - 2x - 4y - 4 = 0$.

31. Given that the circle passes through $(2, 3)$ and has center $(1, 4)$.

    Let the raidus be $r$, then equation of the circle is $(x - 1)^2 + (y - 4)^2 = r^2$. Since it passes through $(2, 3)$

    $(2 - 1)^2 + (3 - 4)^2 = r^2 \Rightarrow r = \sqrt{2}$

    Thus, equation of the circle is $(x - 1)^2 + (y - 4)^2 = 2 \Rightarrow x^2 + y^2 - 2x - 8y + 15 = 0$.

32. Point of intersection of $x + 3y = 0$ and $2x - 7y = 0$ is $(0, 0)$. The other two lines are $x + y + 1 = 0$ and $x - 2y + 4 = 0$.

    Thus, $y = -x - 1$.Putting in $x - 2y + 4 = 0 \Rightarrow x - 2(-x - 1) + 4 = 0 \Rightarrow x = 2, y = -3$. So center is $(2, -3)$.

    Let radius of the circle is $r$. Then the equation of the circle is $(x - 2)^2 + (y + 3)^2 = r^2$

    Since it passes through $(0, 0)$, therefore, $r^2 = 13$.

    Thus, equation of the circle is $x^2 + y^2 - 4x + 6y = 0$.

33. Given that the radius is 5 and center is at $(5, 0)$.

    Thus, equation of the circle is $(x - 5)^2 + y^2 = 5^2 \Rightarrow x^2 + y^2 - 10x = 0$.

34. Let the center be $(h, k)$. Since it lies on the line $x - 2y = 0$, we have $h = 2k$.

    Because the circle passes through $(-1, 2)$ and $(3, -2)$, the distances from the center to these points are equal. So $(h + 1)^2 + (k - 2)^2 = (h - 3)^2 + (k + 2)^2$.

    Simplifying this gives $h - k = 1$. Using $h = 2k$, we get $2k - k = 1$, so $k = 1$ and $h = 2$.

Thus the center is $(2, 1)$. The radius squared is $(2 + 1)^2 + (1 - 2)^2 = 10$.

Therefore, the equation of the circle is $(x - 2)^2 + (y - 1)^2 = 10$.

35. Let the center be $(h, k)$ with $3h + 4k = 7$.

Since the circle passes through $(1, -2)$ and $(4, -3)$, we equate distances $(h - 1)^2 + (k + 2)^2 = (h - 4)^2 + (k + 3)^2$.

Expanding and simplifying gives $6h - 2k - 20 = 0$, which reduces to $3h - k = 10$.

Now solve $3h + 4k = 7$ and $3h - k = 10$.

Subtracting gives $5k = -3$, so $k = -\frac{3}{5}$. Substitute into $3h - k = 10$ $3h + \frac{3}{5} = 10$, so $3h = \frac{47}{5}$ and $h = \frac{47}{15}$.

Thus the center is $\left(\frac{47}{15}, -\frac{3}{5}\right)$.

Now compute radius squared using $(1, -2)$ $\left(\frac{47}{15} - 1\right)^2 + \left(-\frac{3}{5} + 2\right)^2 = \left(\frac{32}{15}\right)^2 + \left(\frac{7}{5}\right)^2 = \frac{1024}{225} + \frac{441}{225} = \frac{1465}{225}$.

So the equation of the circle is $\left(x - \frac{47}{15}\right)^2 + \left(y + \frac{3}{5}\right)^2 = \frac{1465}{225}$.

36. Center of given circle is $(1, 2)$ and radius is $5$.

Let required center be $(h, k)$. Point $(5, 5)$ lies on it $\Rightarrow (h - 5)^2 + (k - 5)^2 = 25$

Collinearity gives $\frac{k-5}{h-5} = \frac{3}{4}$

So $k - 5 = \frac{3}{4}(h - 5)$

Substitute $(h - 5)^2 + \left(\frac{3}{4}(h - 5)\right)^2 = 25 \Rightarrow h = 9$ or $1$

Reject $(1, 2)$, so center is $(9, 8)$.

Eqaution of the cicle is $(x - 9)^2 + (y - 8)^2 = 25$.

37. Perpendicular distance from the center to the tangent will be radius. Let it be $r$.
$r = \frac{|2.1-(-3)-4|}{\sqrt{2^2+(-1)^2}} = \frac{1}{\sqrt{5}}$

Thus, equation of the circle is $(x - 1) + (y + 3)^2 = \frac{1}{5} \Rightarrow x^2 + y^2 - 2x + 6y + \frac{49}{5} = 0$.

38. Since the circle is concentric with the circle $x^2 + y^2 - 4x + 6y - 17 = 0$, therefore, center will be $(2, -3)$.

The circle touches the line $3x - 4y + 7 = 0$, therefore, perpendicular distance from tangent will be radius. Let it be $r$.
$r = \frac{|3.2-4(-3)+7|}{\sqrt{3^2+(-4)^2}} = 5$.

Thus, the equation of the circle is $(x - 2)^2 + (y + 3)^2 = 5^2 \Rightarrow x^2 + y^2 - 4x + 6y - 12 = 0$.

39. Let $(h, k)$ be the center then $(h - 1)^2 + (k - 2)^2 = 25$.

The center is on normal perpendicular to give line. Hence slope will be $-\frac{4}{3}$. Also, slope is $\frac{k-2}{h-1}$.

Thus, $\frac{k-2}{h-1} = -\frac{4}{3} \Rightarrow 4h + 3k = 10 \Rightarrow k = \frac{10-4h}{3}$.

Putting this in the equation of the circle

$(h-1)^2 + \left[\frac{10-4h}{3} - 2\right]^2 = 25 \Rightarrow h^2 - 2h + 1 + \frac{16h^2-32h+16}{9} = 25 \Rightarrow 25h^2 - 50h - 200 = 0$

$h^2 - 2h - 8 = 0 \Rightarrow h = 4, -2, k = -2, 6$.

Thus equation of the circles are $x^2 + y^2 - 8x + 4y - 5 = 0$ and $x^2 + y^2 + 4x - 12y + 15 = 0$.

40. The two normals are $3x - 5y + 2 = 0$ and $x + 2y = 3$. Solving them $3(3-2y) - 5y + 2 = 0 \Rightarrow y = 1 \Rightarrow x = 1$.

    This point of intersection of two normals give the center of the circle. Thus, the center is $(1, 1)$, and it is given that radius of the circle is 5. Hence, the equation is

    $(x-1)^2 + (y-1)^2 = 5^2 \Rightarrow x^2 + y^2 - 2x - 2y - 23 = 0$.

41. Let the center of the circle be $(a, 4)$ and radius be $r$.

    Since the circle touches the $y$-axis at the point $(0, 4)$, the radius equals the distance of the center from the $y$-axis.

    Hence $r = a$

    Equation of the circle $(x-a)^2 + (y-4)^2 = a^2$

    The circle cuts an intercept 6 on the $x$-axis.

    So the points where the circle meets the $x$-axis satisfy $y = 0$.

    Substituting $y = 0$ $(x-a)^2 + 16 = a^2 \Rightarrow (x-a)^2 = a^2 - 16$

    The length of the intercept on the $x$-axis is $2\sqrt{a^2 - 16} = 6 \Rightarrow a = \pm 5$

    Hence the radius is 5 and the center is $(\pm 5, 4)$.

    Therefore the equation of the circle is $(x \mp 5)^2 + (y-4)^2 = 25$.

42. The circle would be in any quadrant and center would be one of $(a, a), (-a, a), (-a, -a)$ and $(a, -a)$.

    Thus, equation of the circle is $(x \mp a)^2 + (y \mp a)^2 = a^2$.

43. Since the circle touches the $y$-axis at origin, therefore, the $x$-axis will be normal. If $r$ is the radius then $(r, 0)$ will be the center,

    Since it passes through $(h, k)$, therefore, $(h-r)^2 + k^2 = r^2 \Rightarrow h^2 + k^2 - 2hr = 0 \Rightarrow r = \frac{h^2+k^2}{2}h$

    Thus, equation of the circle is $(x-r)^2 + y^2 = r^2 \Rightarrow x^2 + y^2 - 2rx = 0 \Rightarrow h(x^2 + y^2) - (h^2 + k^2)x = 0$.

44. Since the circle touches the $x$-axis at origin, therefore, the $y$-axis will be normal. If $r$ is the radius then $(0, r)$ will be the center.

The circle touches the line $4x - 3y + 24 = 0$, so perpendicular distance from the center will be equal to the radius.

$\frac{|4.0 - 3r + 24|}{5} = r \Rightarrow 9(r-8)^2 = 25r^2 \Rightarrow 9r^2 - 144r + 576 = 25r^2 \Rightarrow 16r^2 + 144r - 576 = 0$

$\Rightarrow r^2 + 9r - 36 = 0 \Rightarrow r = 3, -12$

Thus, equation of the circles are $x^2 + y^2 + 24y = 0$ and $x^2 + y^2 - 6y = 0$.

45. Given circle is $x^2 + y^2 = 16 = 4^2$. Thus, parametric equaiton is $x = 4\cos\theta, y = 4\sin\theta$.

46. Let the center of the circle be $(h, k)$ and radius be $r$.

The circle touches the line $2x - y = 1$ at the point $(1, 1)$.

Hence the center lies on the line perpendicular to $2x - y = 1$ through $(1, 1)$.

Slope of $2x - y = 1$ is 2. Therefore the perpendicular slope is $-\frac{1}{2}$.

Equation of the perpendicular line through $(1, 1)$ $y - 1 = \left(-\frac{1}{2}\right)(x-1)$ $x + 2y - 3 = 0$

So the center $(h, k)$ satisfies $h + 2k - 3 = 0$

The radius equals the distance from the center to the line $2x - y - 1 = 0$ and also to the line $2x + y - 4 = 0$

Hence, $\frac{|2h-k-1|}{\sqrt{5}} = \frac{|2h+k-4|}{\sqrt{5}} \Rightarrow |2h - k - 1| = |2h + k - 4|$

Using $h + 2k - 3 = 0$, we get $h = 3 - 2k$

Substituting in the distance equation $|2(3-2k) - k - 1| = |2(3-2k) + k - 4| \Rightarrow k = \frac{7}{8}$

Then $h = 3 - 2\left(\frac{7}{8}\right) \Rightarrow h = \frac{5}{4}$

Radius $r = \frac{|2\left(\frac{5}{4}\right) - \frac{7}{8} - 1|}{\sqrt{5}} \Rightarrow r = \frac{\sqrt{5}}{8}$

Therefore the equation of the circle is $\left(x - \frac{5}{4}\right)^2 + \left(y - \frac{7}{8}\right)^2 = \frac{5}{64}$.

47. Let the center be $(h, k)$ and radius be $r$. Since the center lies on $2x + y = 0$, therefore, $k = -2h$

The circle touches the lines $4x - 3y - 30 = 0$ and $4x - 3y + 10 = 0$

Hence the distances from the center to both lines are equal.

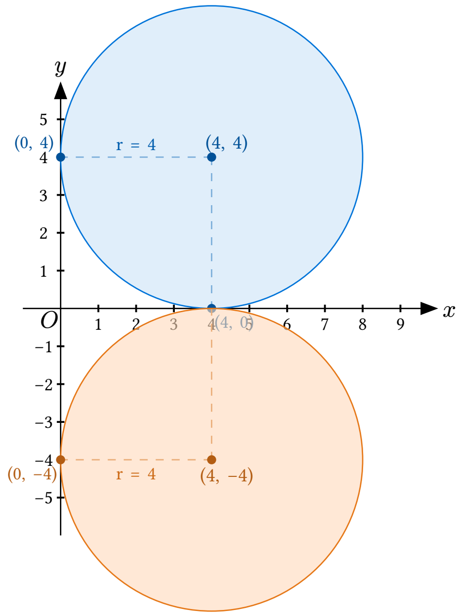
$\frac{|4h-3k-30|}{5} = \frac{|4h-3k+10|}{5} \Rightarrow 4h - 3k = 10$

Using $k = -2h \Rightarrow 4h + 6h = 10 \Rightarrow h = 1, k = -2$

Radius $r = \frac{|4(1)-3(-2)-30|}{5} = 4$

Therefore the equation of the circle is $(x-1)^2 + (y+2)^2 = 16$.

48.

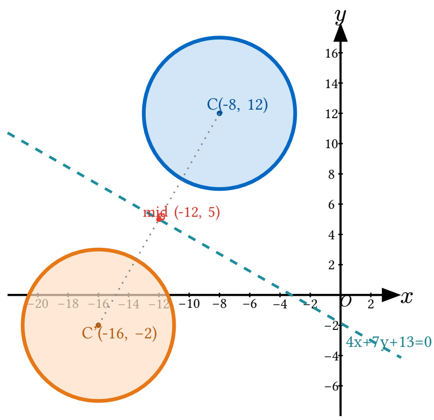



Since the circle touches the coordinate axes and has a radius of 4 units, therefore, its center would be $(4, 4)$ in first quadrant.

Given that $y = 0$ is the line mirror, so the new center would be $(4, -4)$ and the equation of circle will be

$(x - 4)^2 + (y + 4)^2 = 16.$

49.




Center of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ is $(-8, 12)$ and radius is $\sqrt{64 + 144 - 183} = 5.$

Image of the center across this line is given by $x' = x - \frac{2a(ax+by+c)}{a^2+b^2}$ and $y' = y - \frac{2b(ax+by+c)}{a^2+b^2}$

Here $a = 4, b = 7, c = 13$ and $(x, y) = (-8, 12)$.

Now $ax + by + c = 4(-8) + 7(12) + 13 = -32 + 84 + 13 = 65$

Also $a^2 + b^2 = 16 + 49 = 65$. Therefore $x' = -8 - \frac{2.4.65}{65} = -16$ and $y' = 12 - \frac{2.7.65}{65} = -2$

Hence the image circle has center $(-16, -2)$ and radius 5.

Therefore its equation is $(x + 16)^2 + (y + 2)^2 = 25 \Rightarrow x^2 + y^2 + 32x + 4y + 235 = 0$.

50. Center of the circle is $(a, a)$ and radius is $a$. After one complete revolution along $x$-axis will make the new center as $(a + 2\pi a, a)$.

    Thus, new equation is $(x - a - 2\pi a)^2 + (y - a)^2 = a^2$.

51. The center is $(1, 1)$ and radius is 5. The center moves on the line $x - y = 0$, so any new center $(h, k)$ satisfies

    $h - k = 0 \Rightarrow h = k$

    Distance between old and new center is $\sqrt{2}$, so $(h - 1)^2 + (k - 1)^2 = 2$

    Substitute $k = h$ $(h - 1)^2 + (h - 1)^2 = 2$

    $2(h - 1)^2 = 2 \Rightarrow h - 1 = \pm 1$

    **Case I:** $h = 2, k = 2$

    **Case II:** $h = 0, k = 0$

    Thus two circles are possible.

    For $(2, 2)$ equation of circle will be $(x - 2)^2 + (y - 2)^2 = 25$.

    For $(0, 0)$ equation of circle will be $x^2 + y^2 = 25$.

52. Let the circle pass through the origin and have center $(h, k)$ and radius $r$.

    Since it passes through $(0, 0) \Rightarrow h^2 + k^2 = r^2$

    The length of chord cut by the circle on a line is $2\sqrt{r^2 - d^2}$ where $d$ is perpendicular distance from center to the line.

    Given chord length is $\sqrt{2}$.

    So $2\sqrt{r^2 - d^2} = \sqrt{2} \Rightarrow r^2 - d^2 = \frac{1}{2} \Rightarrow h^2 + k^2 - d^2 = \frac{1}{2}$

    Now for line $y = x \Rightarrow x - y = 0$

    Distance from $(h, k)$ is $d_1 = |h - k\frac{|}{\sqrt{2}}$

    So $h^2 + k^2 - \frac{(h-k)^2}{2} = \frac{1}{2}$

    Simplifying gives us $(h + k)^2 = 1$ ...(1)

Now for line $y = -x \Rightarrow x + y = 0$. Distance is $d_2 = |h + k\frac{|}{\sqrt{2}}$

So similarly $h^2 + k^2 - \frac{(h+k)^2}{2} = \frac{1}{2}$

Simplifying gives $(h - k)^2 = 1$ ...(2)

From (1) and (2): $h + k = \pm 1$ and $h - k = \pm 1$

**Case I:** $h + k = 1, h - k = 1$ => $h = 1, k = 0$

**Case II:** $h + k = 1, h - k = -1$ => $h = 0, k = 1$

**Case III:** $h + k = -1, h - k = 1$ => $h = 0, k = -1$

**Case IV:** $h + k = -1, h - k = -1$ => $h = -1, k = 0$

Radius: $r^2 = h^2 + k^2 = 1$. So $r = 1$

Thus the circles are $(x - 1)^2 + y^2 = 1$, $x^2 + (y - 1)^2 = 1$, $x^2 + (y + 1)^2 = 1$, and $(x + 1)^2 + y^2 = 1$.

53. Let $A = 3x + 4y - 15 = 0, \quad B = 3x - 4y - 7 = 0, \quad C = 12x + 5y - 115 = 0$

$\frac{|3x+4y-15|}{5} = \frac{|3x-4y-7|}{5} \Rightarrow 3x + 4y - 15 = \pm(3x - 4y - 7)$

Taking plus sign $3x + 4y - 15 = 3x - 4y - 7 \Rightarrow 8y = 8 \Rightarrow y = 1$

Now use the bisector of $A$ and $C$ $\frac{|3x+4y-15|}{5} = \frac{|12x+5y-115|}{13}$

Substitute $y = 1$ ot get $\frac{|3x-11|}{5} = \frac{|12x-110|}{13} \Rightarrow x = 7$

Hence the incenter is $(7, 1)$.

54. Let the raidus be $r$. The center could be in any quadrant, and thus, center is $(\pm r, \pm r)$.

Since the center lies on $lx + my + n = 0$, therefore, $r = \pm\frac{n}{l+m}$.

Putting this is in $(x \pm r)^2 + (y \pm r)^{=}r^2$ gives us

$(l \pm m)^2(x^2 + y^2) \pm 2n(l \pm m)(x + y) + n^2 = 0.$

55. The smaller circle is $x^2 + y^2 = 4$ so its centre is $(0, 0)$ and radius is 2.

Let the radius of the larger circle be $R$. The distance of the line $x + y = 2$ from the centre is $\sqrt{2}$

Chord length in a circle of radius $r$ at distance $d$ from the centre is $2\sqrt{r^2 - d^2}$

Hence the chord lengths are $2\sqrt{4 - 2} = 2\sqrt{2}$ and $2\sqrt{R^2 - 2}$

Given intercept between the circles is 1

$\Rightarrow \sqrt{R^2 - 2} - \sqrt{2} = 1 \Rightarrow \sqrt{R^2 - 2} = 1 + \sqrt{2} \Rightarrow R^2 = 5 + 2\sqrt{2}$

Therefore, the larger circle is $x^2 + y^2 = 5 + 2\sqrt{2}$

56. The given circle is $x^2 + y^2 - 2x + y = 0$. So the centre is $\left(1, -\frac{1}{2}\right)$ and radius is $\frac{\sqrt{5}}{2}$.

For the point $(3, 2)$, the distance from the centre is $\sqrt{(3 - 1)^2 + \left(2 + \frac{1}{2}\right)^2} = \frac{\sqrt{41}}{2}$

Since $\frac{\sqrt{41}}{2} > \frac{\sqrt{5}}{2}$ the point is exterior to the circle.

The maximum radius of a circle centered at $(3, 2)$ containing the given circle is $\frac{\sqrt{41}}{2} + \frac{\sqrt{5}}{2} = \frac{\sqrt{41}+\sqrt{5}}{2}$

Hence the required circle is $(x-3)^2 + (y-2)^2 = \left(\frac{\sqrt{41}+\sqrt{5}}{2}\right)^2 \Rightarrow (x-3)^2 + (y-2)^2 = \frac{23+\sqrt{205}}{2}$.

57. $C_1 = (0, 2), r_1 = 3$, $C_2 = (-6, -2), r_2 = 3$, and $C_3 = (-3, -6), r_3 = 3$

    $\Delta C_1 C_2 C_3$ is acute, so the minimum enclosing circle of the centers is the circumcircle

    $C = \left(-\frac{31}{18}, -\frac{23}{12}\right) \Rightarrow R = \frac{5\sqrt{949}}{36}$, where $C$ is circumcenter.

    Since each given circle has radius 3, the required minimum radius is $R_{\{min\}} = 3 + \frac{5\sqrt{949}}{36} = \frac{108+5\sqrt{949}}{36}$

    Hence the required circle is $\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(\frac{108+5\sqrt{949}}{36}\right)^2$.

58. The diamter form of the circle is $(x+4)(x-12) + (y-3)(y+1) = 0 \Rightarrow x^2 + y^2 - 8x - 2y - 51 = 0$.

    Putting $x = 0$ for intercept on $y$-axis, we have $y^2 - 2y - 51 = 0$

    Let $y_1$ and $y_2$ be the roots then $y_1 + y_2 = 2$ and $y_1 y_2 = -51$, then

    $|y_1 - y_2| = \sqrt{(y_1 + y_2)^2 - 4y_1 y_2} = 4\sqrt{13}$.

    So the intercept on $y$-axis is $4\sqrt{13}$.

59. One of the diagonals will have endpoints as $(1, 2)$ and $(3, 4)$ and the other will have $(3, 2)$ and $(1, 4)$.

    Thus, the equation of the circle is $(x-1)(x-3) + (y-2)(y-4)$.

    The other diagonal will also give the same equation.

60. Equation of the circle will be $x(x-2) + y(y+4) = 0$.

61. Equation of the circle will be $(x-2)(x+2) + (y+3)(y-4) = 0 \Rightarrow x^2 - 4 + y^2 - y - 12 = 0$

    So center will be $\left(0, \frac{1}{2}\right)$ and radius will be $\sqrt{\frac{1}{4} + 16} = \frac{\sqrt{65}}{2}$.

62. The intercepts are $(3, 0)$ and $(0, 4)$ on $x$ and $y$ axes. Thus, the equation of the circle will be

    $x(x-3) + y(y-4) = 0$.

63. The center of the circles are $(-3, 7)$ and $(2, -5)$. Thus, the equation of the circle is

    $(x+3)(x-2) + (y-7)(y+5) = 0$.

64. One of the diagonals will have endpoints as $(6, 3)$ and $(9, 6)$. Thus equation of the circle will be

$(x-6)(x-9)+(y-3)(y-6)=0$. The other diagonal will also give the same equation.

65. The given lines form a rectangle since there are two pairs of parallel lines.

    From $x-3y=4$ and $3x+y=22$ we get point $A(7,1)$.

    From $x-3y=14$ and $3x+y=22$ we get point $B(8,-2)$.

    From $x-3y=14$ and $3x+y=62$ we get point $C(20,2)$.

    From $x-3y=4$ and $3x+y=62$ we get point $D(19,5)$.

    We take one diagonal $(7,1)$ and $(20,2)$ as diameter to get the equation of the circle as

    $(x-7)(x-20)+(y-1)(y-2)=0$. The other diagonal will also give the same equation.

66. Let $x_1, x_2$ be the roots of $x^2+2x-a^2=0$ then $x_1+x_2=-2$ and $x_1x_2=-a^2$. Similarly, let $y_1, y_2$ be the roots of $y^2+4y-b^2=0$ then $y_1+y_2=-4$ and $y_1y_2=-b^2$.

    Circle whose endpoints will be the diamter $AB$ will be given by $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

    $\Rightarrow x^2-x(x_1+x_2)+x_1x_2+y^2-y(y_1+y_2)+y_1y_2=0$

    Substituting the values from the equations obtained we have the equation as

    $x^2+2x-a^2+y^2+4y-b^2=0 \Rightarrow= (x+1)^2+(y+2)^2=a^2+b^2+5$

    Hence, center is $(-1,-2)$ and radius is $\sqrt{a^2+b^2+5}$.

67. The circle is given by $x^2+y^2-2x+6y-15=0$. Let $(h,k)$ be the other endpoint. Then the equation for the circle with the diameter is given by

    $(x-4)(x-h)+(y-1)(y-k)=0 \Rightarrow x^2-(4+h)+4h+y^2-(1+k)+k=0$

    Comparing coefficients of $x$ and $y$ we have $h=-2$ and $k=-7$.

68. The given lines are $ax+by+c=0$, $ax+by-c=0$, $bx-ay+c=0$ and $bx-ay-c=0$.

    These form a rectangle since each pair is parallel and the two directions are perpendicular.

    Take one pair of opposite vertices by solving $ax+by+c=0$ with $bx-ay+c=0$ and $ax+by-c=0$ with $bx-ay-c=0$.

    Let these points be $P(x_1,y_1)$ and $Q(x_2,y_2)$.

    From symmetry we have $x_2=-x_1$ and $y_2=-y_1$. So the equation of the circle becomes $x^2+y^2=x_1^2+y_1^2$.

    Solving $ax+by+c=0$ and $bx-ay+c=0$ gives $x1=-c\frac{a+b}{a^2+b^2}$ and $y1=-c\frac{b-a}{a^2+b^2}$.

So $x_1^2 + y_1^2 = \frac{c^2((a+b)^2+(b-a)^2)}{(a^2+b^2)^2}$.

This simplifies to $x_1^2 + y_1^2 = \frac{2c^2}{a^2+b^2}$.

Hence the equation of the circumcircle is $x^2 + y^2 = \frac{2c^2}{a^2+b^2}$.

69. Let the equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ whose center is $(-g, -f)$, which lies on $3x + 4y = 7$. Thus,

$-3g - 4f = 7$. Since the circle passes through $(1, -2)$ and $(4, -3)$, therefore,

$2g - 4f + c = -5$ and $8g - 6f + c = 25$. From these two equations we have $-3g + f = 10$

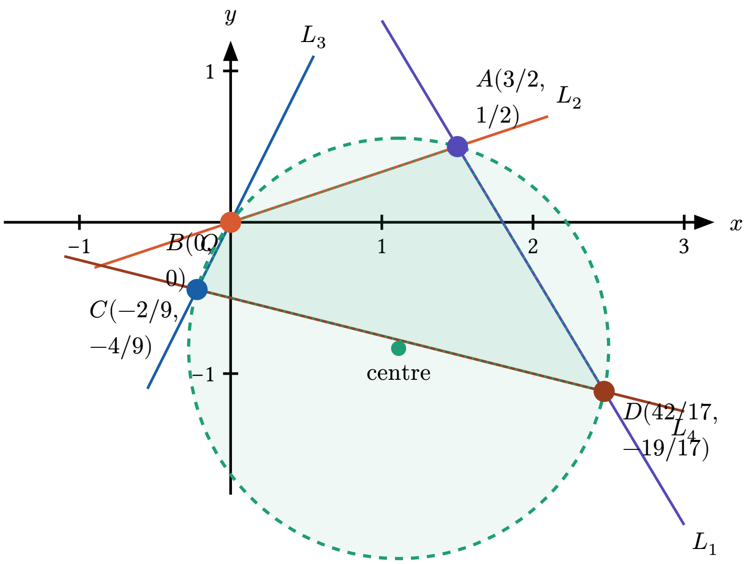
Thus, $f = \frac{3}{5}, g = -\frac{47}{15}$ and $c = \frac{11}{3}$ and now it is trivial to find the equation.

70. The line $3x + 4y = 12$ meets the axes at $(4, 0)$ and $(0, 3)$. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Since it passes through origin, therefore, $c = 0$.

For $(4, 0)$m the equation becomes $16 + 8g = 0 \Rightarrow g = -2$ and for $(0, 3)$ the equation is $9 + 6f = 0 \Rightarrow f = -\frac{3}{2}$. Thus, we have found the equation of the circle as $x^2 + y^2 - 4x - 3y = 0$.

71.




The given lines are $5x + 3y = 9$, $x = 3y$, $2x = y$ and $x + 4y + 2 = 0$.

From $5x + 3y = 9$ and $x = 3y$ we get $A\left(\frac{3}{2}, \frac{1}{2}\right)$.

From $x = 3y$ and $2x = y$ we get $B(0, 0)$.

From $2x = y$ and $x + 4y + 2 = 0$ we get $C\left(-\frac{2}{9}, -\frac{4}{9}\right)$.

We take the general circle $x^2 + y^2 + gx + fy + c = 0$.

Substitute point $B(0,0)$ and get $c = 0$.

Substitute point $A(\frac{3}{2}, \frac{1}{2})$.

This gives $\frac{5}{2} + 3\frac{g}{2} + \frac{f}{2} = 0$ so $3g + f = -5$.

Substitute point $C(-\frac{2}{9}, -\frac{4}{9})$.

This gives $\frac{20}{81} - 2\frac{g}{9} - 4\frac{f}{9} = 0$.

Multiply by 81 to get $20 - 18g - 36f = 0$ so $9g + 18f = 10$.

From $3g + f = -5$ we get $f = -5 - 3g$.

Substitute into $9g + 18f = 10$.

This gives $9g + 18(-5 - 3g) = 10$.

So $9g - 90 - 54g = 10$ which gives $-45g = 100$.

Thus, $g = -\frac{20}{9}$ and $f = \frac{5}{3}$.

Hence the circle is $x^2 + y^2 - \frac{20}{9}x + \frac{5}{3}y = 0$.

72. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since it passes through $(1,2)$ and $(3,4)$, therefore,

$5 + 2g + 4f + c = 0$ ...(1) and $25 + 6g + 8f + c = 0$ ...(2)

From these two equations we have $g + f + 5 = 0$ ...(3)

Since the circle touches the line $3x + y - 3 = 0$ so perpendicular distance from center would be radius. Thus,
$\frac{|-3g-f-3|}{\sqrt{10}} = \sqrt{g^2 + f^2 + c} \Rightarrow (3g + f + 3)^2 = 10(g^2 + f^2 + 5 + 2g + 4f)$[from (1)]

$(2g - 5 + 3)^2 = 10[g^2 + (g + 5)^2 + 5 + 2g - 4g - 29]$[Putting the value of $f$ from (3)]

$\Rightarrow g = -4, -\frac{3}{2} \Rightarrow f = -1, -\frac{7}{2} \Rightarrow c = 7, 12$

Thus, equation of the circles are $x^2 + y^2 - 8x - 2y + 7 = 0$ and $x^2 + y^2 - 3x - 7y + 12 = 0$.

73. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$, so the center is $(-g, -f)$ and radus us $\sqrt{g^2 + f^2 - c}$

Since the circle touches the $x$-axis, therefore, $g^2 - c = 0 \Rightarrow c = g^2 \left[\because \sqrt{g^2 + f^2 - c = |-f|}\right]$ ...(1)

Also, the circle touches $4x - 3y + 4 = 0 \Rightarrow \frac{|-4g+3f+4|}{5} = \sqrt{g^2 + f^2 + c} = |f|$[from (1)]

$\Rightarrow -4g + 3f + 4 = \pm f \Rightarrow 2g + f = 2$ ...(2) and $g - 2f = 1$ ...(3)

Also, given that the center lies on $x - y - 1 = 0 \Rightarrow -g + f = 1$ ...(4)

Thus, $g = \frac{1}{3}, f = \frac{4}{3}$ and $f = -2, g = -3$, which lies in first quadrant. Thus, $c = \frac{1}{9}$.

Hence, the equation of the cirlce is $9(x^2 + y^2) + 6x + 24y + 1 = 0$.

74. Let the circle be $x^2 + y^2 + gx + fy + c = 0$. Substituting point $(1, 0)$ gives

    $1 + g + c = 0$ so $g + c = -1$.

    Substituting point $(0, 1)$ gves $1 + f + c = 0$ so $f + c = -1$.

    Substituting point $(1, -2)$ gives $1 + 4 + g - 2f + c = 0$ so $g - 2f + c = -5$.

    From $g + c = -1$ we get $g = -1 - c$. From $f + c = -1$ we get $f = -1 - c$.

    Substitute into $g - 2f + c = -5$.

    This gives $(-1 - c) - 2(-1 - c) + c = -5$. So $-1 - c + 2 + 2c + c = -5$.

    This simplifies to $1 + 2c = -5$ so $c = -3$. Then $g = 2$ and $f = 2$.

    Hence the equation of the circle is $x^2 + y^2 + 2x + 2y - 3 = 0$.

75. Let the circle be $x^2 + y^2 + gx + fy + c = 0$.

    Substituting point $(0, 0)$ gives $c = 0$.

    Substituting point $(a, 0)$ gives $a^2 + ga = 0$ so $g = -a$.

    Substituting point $(0, b)$ gives $b^2 + fb = 0$ so $f = -b$.

    Hence, the equation of the circle is $x^2 + y^2 - ax - by = 0$.

76. Let the circle be $x^2 + y^2 + gx + fy + c = 0$. Since it passes through the origin, we get $c = 0$.

    So the equation becomes $x^2 + y^2 + gx + fy = 0$.

    Now we consider the intercept on the positive $x$ axis. Putting $y = 0$ gives $x^2 + gx = 0$ which gives $x(x + g) = 0$.

    So the intercept points are $x = 0$ and $x = -g$.

    The length of the chord on the positive $x$ axis is 4. Hence, $-g = 4$ so $g = -4$.

    Now we consider the intercept on the positive $y$ axis. Putting $x = 0$ gives $y^2 + fy = 0$ which gives $y(y + f) = 0$.

    So the intercept points are $y = 0$ and $y = -f$.= The length of the chord on the positive $y$ axis is 6.

    Hence, $-f = 6$ so $f = -6$. Therefore, the equation of the circle is $x^2 + y^2 - 4x - 6y = 0$.

77. The given lines are $y = x$, $y = 2x$ and $y = 3x + 2$.

    From $y = x$ and $y = 2x$ we get $A(0, 0)$.

    From $y = x$ and $y = 3x + 2$ we get $B(-1, -1)$.

    From $y = 2x$ and $y = 3x + 2$ we get $C(-2, -4)$.

    Let the circle be circle $x^2 + y^2 + gx + fy + c = 0$.

Substituting point $A(0,0)$ and get $c=0$.

Substituting point $B(-1,-1)$ gives $1+1-g-f=0$ so $g+f=2$.

Substituting point $C(-2,-4)$ gives $4+16-2g-4f=0$ so $g+2f=10$.

From $g+f=2$ we get $g=2-f$.

Substituting into $g+2f=10$ gives $2-f+2f=10$ so $f=8$.

Then $g=-6$.

Hence, the equation of the circumcircle is $x^2+y^2-6x+8y=0$.

78. The given sides of the triangle are $7x-y+11=0$, $x+y-15=0$ and $7x+17y+65=0$.

From $7x-y+11=0$ and $x+y-15=0$ we get $A\left(\frac{1}{2},\frac{29}{2}\right)$.

From $x+y-15=0$ and $7x+17y+65=0$ we get $B(-10,25)$.

From $7x+17y+65=0$ and $7x-y+11=0$ we get $C\left(-\frac{19}{3},-\frac{100}{3}\right)$.

If the side lengths opposite $A,B,C$ are $a,b,c$ then the incenter is $\frac{ax_1+bx_2+cx_3}{a+b+c}$ and $\frac{ay_1+by_2+cy_3}{a+b+c}$.

Length $a=BC=5\sqrt{85}$. Length $b=CA=43\sqrt{5}$. Length $c=AB=3\sqrt{85}$.

After simplification the incenter is $(-3,11)$.

Now find the radius which is the perpendicular distance from the incenter to any side.

Distance to $x+y-15=0$ is $\frac{|-3+11-15|}{\sqrt{2}}=\frac{7}{\sqrt{2}}$.

Hence, the equation of the incircle is $(x+3)^2+(y-11)^2=\frac{49}{2}$.

79. Let the circle be $x^2+y^2+gx+fy+c=0$. Since it passes through the origin, we get $c=0$.

So the equation becomes $x^2+y^2+gx+fy=0$.

Now we consider the line $3x=4y$ which is $3x-4y=0$.

The perpendicular distance from the center $\left(-\frac{g}{2},-\frac{f}{2}\right)$ to this line is

$\frac{|3(-\frac{g}{2})-4\left(-\frac{f}{2}\right)|}{5}=\frac{|-3g+4f|}{10}$.

Since the circle cuts off a chord of length 1 on this line, we use

$1=2\sqrt{r^2-d^2}$. So $r^2-d^2=\frac{1}{4}$.

Now $r^2=\frac{g^2+f^2}{4}$. So $\frac{g^2+f^2}{4}-\left(\frac{(-3g+4f)^2}{100}\right)=\frac{1}{4}$.

Similarly for the line $4x=3y$ which is $4x-3y=0$. Distance from center is $\frac{|-4g+3f|}{10}$.

So $\frac{g^2+f^2}{4}-\left(\frac{(-4g+3f)^2}{100}\right)=\frac{1}{4}$.

Now subtract the two equations. This gives $(-3g+4f)^2 = (-4g+3f)^2$.

So either $-3g+4f = -4g+3f$ or $-3g+4f = 4g-3f$. First case gives $g+f = 0$. Second case gives $g = f$.

Now substitute each case. For $g+f=0$ we get $g=-f$.

Substitute into equation and solve to get $g=1$ and $f=-1$.

For $g=f$ we substitute and get $g=-1$ and $f=-1$.

Hence the required circles are $x^2+y^2+x-y=0$ and $x^2+y^2-x-y=0$.

80. Common chord of the circles is $x^2+y^2-4x-5-(x^2+y^2+8y+7)=0 \Rightarrow x+2y+3=0$

Equation of such a circle is $x^2+y^2-4x-5+k(x+2y+3)=0 \Rightarrow x^2+y^2-(4-k)x+2ky-3k-5=0$.

Its center is $\left(\frac{4-k}{2}, -k\right)$. If $x+2y+3=0$ is diameter then $\frac{4-k}{2}-2k+3=0 \Rightarrow k=2$.

Thus, equation of the circle is $x^2+y^2-2x+4y+1=0$.

81. Equation of any circle passing through the point of intersection of the given circle and the given chord is $x^2+y^2-a^2+k(x\cos\alpha+y\sin\alpha-p)=0$.

Center of this circle is $\left(-\frac{k\cos\alpha}{2}, -\frac{k\sin\alpha}{2}\right)$.

Since $x\cos\alpha+y\sin\alpha-p=0$ is the diameter of this circle the center will lie on this line, therefore,

$-\frac{k\cos\alpha}{2}\cos\alpha-\frac{k\sin\alpha}{2}\sin\alpha-p=0 \Rightarrow k=-2p$

Thus, the equation of the circle becomes $x^2+y^2-a^2-2p(x\cos\alpha+y\sin\alpha-p)=0$.

82. Clearly $x^2+y^2-4=0$ is the equation of a circle with center at origin and radius 2.

Also line $y=mx+2\sqrt{1+m^2}$ is the equation of the line which touches the circle for all values for $m$.

Let $P$ be the point of contact of the circle and the line. Clearly, given equation is the equation of circles passing through the point of contact of the given circle and the given line. Any two circles of this family touch each other at $P$.

83. Equation of the line joining the points $(x_1, y_1)$ and $(x_2, y_2)$ is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Also equation of the circle with $(x_1, y_1)$ and $(x_2, y_2)$ as endpoints of the diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$.

Equation of any circle passing through the point of intersection of the above circle and line is given by $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$.

Putting $\lambda = 0$ gives $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$, which is the diameter form of the equation of the circle.

84. Equation of any circle through the point of intersection $P$ and $Q$ of the line and the circle is $x^2 + y^2 + ax + by + c + \lambda(Ax + By + C) = 0$

$\Rightarrow x^2 + y^2 + (a + \lambda A)x + (b + \lambda B)y + c + \lambda C = 0.$

Similarly for other pair of line and circle $x^2 + y^2 + (a' + \mu A')x + (b' + \mu B')y + c' + \mu C' = 0$

If the two circles are the same then the points $P, Q, R$ and $S$ will be concyclic.

Comparing coefficients $1 = \frac{a+\lambda A}{a'+\mu A'} = \frac{b+\lambda B}{b'+\mu B'} = \frac{c+\lambda C}{c'+\mu C'}$

Thus, $a - a' + \lambda A - \mu A' = 0$, $b - b' + \lambda B - \mu B' = 0$, and $c - c' + \lambda C - \mu C' = 0$.

Eliminating $\lambda$ and $-\mu$ and writing in discriminant form we have

$\begin{vmatrix} a-a' & A & A' \\ b-b' & B & B' \\ c-c' & C & C' \end{vmatrix} = \begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$

85. Equation of any circle possing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$S \equiv (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ ...(1)

Let the fixed circle be $S' = x^2 + y^2 + 2gx + 2fy + c = 0$ ...(2)

Equation of the chord of intersection of circles (1) and (2) will be $S - S' = 0$

$\Rightarrow -(x_1 + x_2 + 2g)x - (y_1 + y_2 + 2f)y + x_1x_2 + y_1y_2 - c + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ ...(3)

Clearly this line passes through the point of intersection of two fixed lines $-(x_1 + x_2 + 2g)x - (y_1 + y_2 + 2f)y + x_1x_2 + y_1y_2 - c = 0$ and $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$, which is a fixed point.

86. Given circle is $x^2 + y^2 - a^2 = 0$ ...(1)

Since $PQ$ and $PR$ are tangents to the circle (1), therefore $QR$ will be the chord of contact of point $(x_1, y_1)$, and hence, equaiton of $QR$ will be

$xx_1 + yy_1 - a^2 = 0$ ...(2)

Equation of any circle through the point of intersection $Q$ and $R$ of (1) and (2) is

$x^2 + y^2 - a^2 + k(xx_1 + yy_1 - a^2) = 0$ ...(3)

Circle (3) will be circumcircle of $\triangle PQR$ if circle (3) passes through the point $P(x_1, y_1)$ i.e.

$x_1^2 + y_1^2 - a^2 + k(x_1^2 + y_1^2 - a^2) = 0 \Rightarrow k = -1$

Hence, required circle is $x^2 + y^2 - xx_1 - yy_1 = 0$.

87. Given circles are $x^2 + y^2 - 6x + 2y + 4 = 0$ ...(1) and $x^2 + y^2 + 2x - 4y - 6 = 0$ ...(2) and given line is $x - y = 0$ ...(3).

    Equation of any circle passing through the point of intersection of circles (1) and (2) is

    $x^2 + y^2 - 6x + 2y + 4 + k(x^2 + y^2 + 2x - 4y - 6) = 0 \Rightarrow (1 + k)x^2 + (1 + k)y^2 - 2(3 - k)x + 2(1 - 2k)y + 4 - 6k = 0$

    Its center is $\left(\frac{3-k}{1+k}, \frac{2k-1}{1+k}\right)$. Since it lies on the line (3), therefore,

    $\frac{3-k}{1+k} - \frac{2k-1}{1+k} = 0 \Rightarrow k = \frac{4}{3}$.

    Thus, required equation is $x^2 + y^2 - \frac{10}{7}x - \frac{10}{7}y - \frac{12}{7} = 0$.

88. Let $S_1 = x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1) and $S_2 = x^2 + y^2 + 2g'x + 2f'y + c' = 0$ ...(2)

    Now equation of common chord of the circles is $S_1 - S_2 = 0 \Rightarrow 2(g - g')x + 2(f - f')y + c - c' = 0$ ...(3)

    Since cirlce (1) bisects the circumference of the circle (2), therefore, common chord will be the diameter of the circle (2) and hence center will be $(-g', -f')$ of circle (2) will lie on the line (3)

    $\Rightarrow -2(g - g')g' - 2(f - f')f' + c - c' = 0 \Rightarrow 2g'(g - g') + 2f'(f - f') = c - c'$.

89. The given circles are $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$.

    The family of circles passing through their points of intersection is $S_1 + \lambda S_2 = 0$.

    So the required circle is $(x^2 + y^2 - 2x - 4y - 4) + \lambda(x^2 + y^2 - 10x - 12y + 40) = 0$.

    This simplifies to $(1 + \lambda)(x^2 + y^2) + (-2 - 10\lambda)x + (-4 - 12\lambda)y + (-4 + 40\lambda) = 0$.

    $\Rightarrow x^2 + y^2 + \frac{-2-10\lambda}{1+\lambda}x + \frac{-4-12\lambda}{1+\lambda}y + \frac{-4+40\lambda}{1+\lambda} = 0$.

    Comparing with $x^2 + y^2 + gx + fy + c = 0$.

    So $g = \frac{-2-10\lambda}{1+\lambda}$, $f = \frac{-4-12\lambda}{1+\lambda}$, and $c = \frac{-4+40\lambda}{1+\lambda}$.

    The radius condition is $g^2 + f^2 - c = 16$. After solving we get $\lambda = 1$.

    Then the equation becomes $2(x^2 + y^2) - 12x - 16y + 36 = 0 \Rightarrow x^2 + y^2 - 6x - 8y + 18 = 0$.

90. The given circles are $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 8x - 6y + 23 = 0$.

    The common chord is obtained by subtracting the two equations.
    So we get $(x^2 + y^2 - 6x - 4y + 9) - (x^2 + y^2 - 8x - 6y + 23) = 0$.

    This simplifies to $2x + 2y - 14 = 0$ or $x + y - 7 = 0$.

From $x^2 + y^2 - 8x - 6y + 23 = 0$ the center is $(4, 3)$.

Substitute $(4, 3)$ into $x + y - 7 = 0$. We get $4 + 3 - 7 = 0$.

So the common chord passes through the center of the second circle.

The radius of the second circle is $r^2 = 16 + 9 - 23 = 2$ so $r = \sqrt{2}$.

The perpendicular distance from the center $(4, 3)$ to the chord $x + y - 7 = 0$ is $\frac{|4+3-7|}{\sqrt{2}} = 0$.

Hence, the chord passes through the center, so it is a diameter.

Therefore, the length of the chord is $2r = 2\sqrt{2}$.

91. The given circles are $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.

    The common chord is obtained by subtracting the equations.

    So we get $(x^2 + y^2 + 2x + 3y + 1) - (x^2 + y^2 + 4x + 3y + 2) = 0. \Rightarrow -2x - 1 = 0$ or $x = -\frac{1}{2}$.

    Now for a circle with diameter along a line, we use the fact that its center lies on the perpendicular bisector of the chord.

    The midpoint of the chord lies on the line joining the centers of the two given circles.

    The centers are $\left(-1, -\frac{3}{2}\right)$ and $\left(-2, -\frac{3}{2}\right)$. So the line joining centers is $y = -\frac{3}{2}$.

    The midpoint of the chord is intersection of $x = -\frac{1}{2}$ and $y = -\frac{3}{2}$.

    So the center is $\left(-\frac{1}{2}, -\frac{3}{2}\right)$. Substitute $x = -\frac{1}{2}$ in first circle.

    Then $\left(\frac{1}{4}\right) + y^2 - 1 + 3y + 1 = 0$. This gives $y^2 + 3y + \frac{1}{4} = 0$.

    Solve to get $y = \frac{-3 \pm 2\sqrt{2}}{2}$.

    So the radius squared is $r^2 = \left(\sqrt{2}\right)^2 = 2$.

    Hence, the equation of the circle is $\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{3}{2}\right)^2 = 2$.

92. The given circle is $x^2 + y^2 - 2ax = 0$ and the chord is $y = mx$.

    Substitute $y = mx$ into the circle.

    This gives $x^2 + m^2x^2 - 2ax = 0 \Rightarrow (1 + m^2)x^2 - 2ax = 0$.

    So the points of intersection are $x = 0$ and $x = 2\frac{a}{1+m^2}$.

    Thus the points are $(0, 0)$ and $\left(2\frac{a}{1+m^2}, 2a\frac{m}{1+m^2}\right)$.

    These are the endpoints of the chord.

    The equation of the circle with this chord as diameter is $\left(2\frac{a}{1+m^2}, 2a\frac{m}{1+m^2}\right)$.

    This gives $x\left(x - 2\frac{a}{1+m^2}\right) + y\left(y - 2a\frac{m}{1+m^2}\right) = 0 \Rightarrow (1 + m^2)(x^2 + y^2) - 2a(x + my) = 0$.

93. The given circles are $x^2 + y^2 - 6x + 2y + 4 = 0$ and $x^2 + y^2 + 2x - 4y - 6 = 0$.

The family of circles passing through their points of intersection is $S_1 + \lambda S_2 = 0$. So the required circle is

$(x^2 + y^2 - 6x + 2y + 4) + \lambda(x^2 + y^2 + 2x - 4y - 6) = 0$.

$\Rightarrow (1 + \lambda)(x^2 + y^2) + (-6 + 2\lambda)x + (2 - 4\lambda)y + (4 - 6\lambda) = 0$.

$\Rightarrow x^2 + y^2 + \frac{-6+2\lambda}{1+\lambda}x + \frac{2-4\lambda}{1+\lambda}y + \frac{4-6\lambda}{1+\lambda} = 0$.

The center is $\left(\frac{6-2\lambda}{2(1+\lambda)}, \frac{-2+4\lambda}{2(1+\lambda)}\right)$.

Since the center lies on $y = x$, equate the coordinates. So $(6 - 2\lambda) = (-2 + 4\lambda)$.

This gives $8 = 6\lambda$ so $\lambda = \frac{4}{3}$.

Then $1 + \lambda = \frac{7}{3}$. So the equation becomes

$\left(\frac{7}{3}\right)(x^2 + y^2) + \left(-\frac{10}{3}\right)x + \left(-\frac{10}{3}\right)y - 4 = 0 \Rightarrow 7(x^2 + y^2) - 10x - 10y - 12 = 0$.

94. The given equation is $x^2 + y^2 + 2(3 + p)x + 2(3 - p)y + 4 = 0$.

This is of the form $x^2 + y^2 + gx + fy + c = 0$ so it represents a circle for all values of $p$.

$\Rightarrow x^2 + y^2 + 6x + 6y + 4 + 2p(x - y) = 0$.

For fixed points, the equation must be satisfied for all values of $p$.

So the coefficient of $p$ must be zero and the remaining part must also be zero.

Thus we get $x - y = 0$ and $x^2 + y^2 + 6x + 6y + 4 = 0$.

From $x - y = 0$ we get $y = x$.

Substitute into the second equation $x^2 + x^2 + 6x + 6x + 4 = 0 \Rightarrow 2x^2 + 12x + 4 = 0$.

Solving gives $x = -3 \pm \sqrt{7}$.

Since $y = x$, the fixed points are $\left(-3 + \sqrt{7}, -3 + \sqrt{7}\right)$ and $\left(-3 - \sqrt{7}, -3 - \sqrt{7}\right)$.

95. The given circles are $x^2 + y^2 - 4a^2 = 0$ and $x^2 + y^2 - 2x - 4y + 4 = 0$.

The family of circles through their intersection is $(x^2 + y^2 - 4a^2) + \lambda(x^2 + y^2 - 2x - 4y + 4) = 0$.

This gives $(1 + \lambda)(x^2 + y^2) - 2\lambda x - 4\lambda y + (-4a^2 + 4\lambda) = 0$.

The center is $\left(\frac{\lambda}{1+\lambda}, 2\frac{\lambda}{1+\lambda}\right)$.

Since the circle touches $x + 2y = 0$, the distance from center equals radius.

This gives $\frac{4a^2-4\lambda}{1+\lambda} = 0$ so $\lambda = a^2$.

Hence, the required circle is $(1 + a^2)(x^2 + y^2) - 2a^2x - 4a^2y + 4a^2(1 - a^2) = 0$.

96. The given circle is $x^2 + y^2 - x - y = 0$ and the line is $x + y = 1$.

The family of circles passing through their intersection points is $x^2 + y^2 - x - y + \lambda(x + y - 1) = 0$.

This gives $x^2 + y^2 + (-1 + \lambda)x + (-1 + \lambda)y - \lambda = 0$.

Since the circle passes through $(1, 1)$, substitute it.

So $1 + 1 + (-1 + \lambda) + (-1 + \lambda) - \lambda = 0 \Rightarrow \lambda = -2$.

Hence, the required circle is $x^2 + y^2 - 3x - 3y + 2 = 0$.

97. The given circle is $x^2 + y^2 = a^2$ and the line is $px + qy - 1 = 0$.

Let $(x_1, y_1)$ and $(x_2, y_2)$ be the endpoints of this chord.

Equation of the circle is $x^2 + y^2 - (x_1 + x_2)x - (y_1 + y_2)y + (x_1x_2 + y_1y_2) = 0$.

Now $(x_1 + x_2, y_1 + y_2)$ is twice the midpoint of the chord.

The midpoint is the foot of the perpendicular from the center $(0, 0)$ to the line.

So midpoint is $\left(\frac{p}{p^2+q^2}, \frac{q}{p^2+q^2}\right)$.

Hence, $x_1 + x_2 = 2\frac{p}{p^2+q^2}$ and $y_1 + y_2 = 2\frac{q}{p^2+q^2}$.

Also both points satisfy $px + qy = 1$. So $p(x_1 + x_2) + q(y_1 + y_2) = 2$.

Thus $x_1x_2 + y_1y_2 = \frac{1}{p^2+q^2}$. Substitute in the diameter form.

Hence, the required circle is $x^2 + y^2 - \left(\frac{2p}{p^2+q^2}\right)x - \left(\frac{2q}{p^2+q^2}\right)y + \frac{1}{p^2+q^2} = 0$.

98. The given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the external point is $A(\alpha, \beta)$.

Let $P$ and $Q$ be the points of contact of tangents from $A$. The chord of contact of $A$ with respect to the circle is $T = 0$.

So the equation of chord $PQ$ is $x\alpha + y\beta + g(x + \alpha) + f(y + \beta) + c = 0$.

The circumcircle of $\triangle PQR$ where $R$ is the center of the given circle is obtained by combining $S = 0$ and $T = 0$.

So its equation is $S + \lambda T = 0$.

Since it passes through $A(\alpha, \beta)$, we substitute it. Then $S_1 + \lambda T_1 = 0$.

Here $T_1 = S_1$. So we get $S_1(1 + \lambda) = 0$ which gives $\lambda = -1$.

Hence, the required circle is $S - T = 0$.

So the equation is $x^2 + y^2 + 2gx + 2fy + c - [x\alpha + y\beta + g(x + \alpha) + f(y + \beta) + c] = 0$.

$\Rightarrow x^2 + y^2 + (g - \alpha)x + (f - \beta)y - (g\alpha + f\beta) = 0$.

99. Substituting the value of $y = \frac{1}{4}(3x - c)$ in the equation of the circle gives us

$x^2 + \frac{1}{16}(3x - c)^2 - 4x - \frac{8.1}{4}(3x - c) - 5 = 0 \Rightarrow 25x^2 - 2(80 + 3c)x + c^2 + 32c - 80 = 0$

The given line and circle will intersect if the above quadratic equation's roots are real i.e. discrimininate $> 0$

$\Rightarrow 4(80 + 3c)^2 - 100(c^2 + 32c - 80) > 0 \Rightarrow c^2 + 20c - 525 < 0 \Rightarrow -35 < c < 15$.

100. Center of the circle is $\left(-\frac{3}{2}, \frac{1}{2}\right)$ and radius is $\frac{5}{\sqrt{2}}$. Let $l$ be the length of the perpendicular from the center to the given line then

$l = \frac{|4(-\frac{3}{2}) - \frac{3}{2} - 5|}{5} = \frac{5}{2}$

Hence, length of the chord is $2\sqrt{\frac{25}{2} - \frac{25}{4}} = \frac{5}{2}$.

101. Center of the circle is $(0, 0)$ and its perpendicular distance from the line is $\frac{|a\sqrt{2}|}{\sqrt{2}} = a$, which is equal to the radius of the circle.

Hence, the given circle touches the given line. Let $(\alpha, \beta)$ be the point of contact. Then equation of tangent is given by $\alpha x + \beta y - a^2 = 0$

Comparing the coefficients with the given equation of the line we have

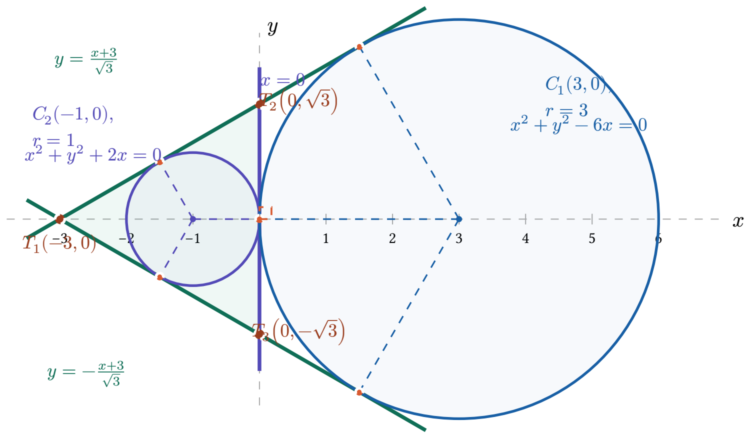
$\frac{\alpha}{1} = \frac{\beta}{-}1 = -\frac{a^2}{a\sqrt{2}} \Rightarrow (\alpha, \beta) = \left(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$.

102. Center of the given circle is $(3, -2)$ and radius is 5. Equation of any line parallel to given line is $4x + 3y + k = 0$.

Since this point is tangent to the given circle, therefore,

$\frac{|4.3 - 3.2 + k|}{5} = 5 \Rightarrow |6 + k| = 25 \Rightarrow k = 19, -31$.

103.




Given circles are $S_1 = x^2 + y^2 - 6x = 0$ ...(1) and $S_2 = x^2 + y^2 + 2x = 0$ ...(2).

Let $A$ and $B$ are the centers and $r_1$ and $r_2$ the radii of $S_1$ and $S_2$ respectively.

$A = (3, 0), B = (-1, 0), r_1 = 3, r_2 = 1 \therefore r_1 + r_2 = 4$. Hence two circles touch each other. Thus, there will be three common tangents.

Equation of chord is given by $S_1 - S_2 = x = 0$, when $x = 0, y = 0$. Thus, $x = 0$ is a common tangent.

Let $y = mx + c$ be a common tangent to the given circles then

$\frac{|3m + c|}{\sqrt{1 + m^2}} = 3$ and $\frac{|-m + c|}{\sqrt{1 + m^2}} = 1$

Solving these two equations gives us $c = \pm\sqrt{3}$ and $m = \pm\frac{1}{\sqrt{3}}$

Thus, common tangents are $x = 0, y = \frac{x}{\sqrt{3}} + \sqrt{3}$, and $y = -\frac{x}{\sqrt{3}} - \sqrt{3}$

Let $P, Q, R$ be points of intersections of these three lines then $P = \left(0, \sqrt{3}\right), Q = (-3, 0)$ and $R = \left(0, -\sqrt{3}\right)$.

It is trivial to prove that $\triangle\, PQR$ is an equilateral triangle.

104. Give that the biggest circle is $x^2 + y^2 = 1 \qquad ...(1)$. Since the radiio of the circles are in A.P. let the commond difference be $d$.

Thus, other two circles will be $x^2 + y^2 = (1-d)^2 \qquad ...(2)$ and $x^2 + y^2 = (1-2d)^2 \qquad ...(3)$

Given line is $y = x + 1$. Putting in (1) gives us $x^2 + (x+1)^2 = 1 \Rightarrow x = 0, -1$

Similarly with (2) we have $x^2 + (x+1)^2 = \left(1 - d^2\right) \Rightarrow 2x^2 + 2x + 2d - d^2 = 0$

Since the points are real and distinct, therefore, $4 - 8\left(2d - d^2\right) > 0 \Rightarrow 2d^2 - 4d + 1 > 0$

$1 - \frac{1}{\sqrt{2}} > d > 1 + \frac{1}{\sqrt{2}}$

Similarly with (3) we have $x^2 + (x+1)^2 = (1-2d)^2$. Proceeding similarly we obtain

$\frac{2-\sqrt{2}}{4} > d > \frac{2+\sqrt{2}}{4}$. However, $d < 1$.

Thus, we have $0 < d < \frac{2-\sqrt{2}}{4}$.

105. Give $4l^2 - 5m^2 + 6l + 1 = 0 \qquad ...(1)$ and line is $lx + my + 1 = 0 \qquad ...(2)$

Let the center of the circle be $(\alpha, \beta)$ with radius $a$. Then

$\frac{|l\alpha + m\beta + 1|}{\sqrt{l^2 + m^2}} = a$

$\Rightarrow l^2\alpha^2 + m^2\beta^2 + 1 + 2lm\alpha\beta + 2l\alpha + 2m\beta = a^2l^2 + a^2m^2$

$\Rightarrow \left(\alpha^2 - a^2\right)^2 + (\beta^2 - a^2)m^2 + 2lm\alpha\beta + 2\alpha l + 2m\beta + 1 = 0$

Comparing this with (1)

$\alpha^2 - a^2 = 4,\ \beta^2 - a^2 = -5,\ \alpha = 3,\ \beta = 0$. Thus, $a = \sqrt{5}$.

Hence, the circle has center $(3, 0)$ and radius $\sqrt{5}$.

106. Given cicle is $x^2 + y^2 - 4x - 6y + 9 = 0 \qquad ...(1)$. Its center is $C(2, 3)$ and its radius is 2.

Let $OP$ be a tangent and let $y$-axis(which is a tanegent) touch the circle at $N$. Then $\angle POX$ will be minimum when $OP$ is tangent to the circle.

Let $\angle POX = \theta$ then $\angle LCP = \theta$

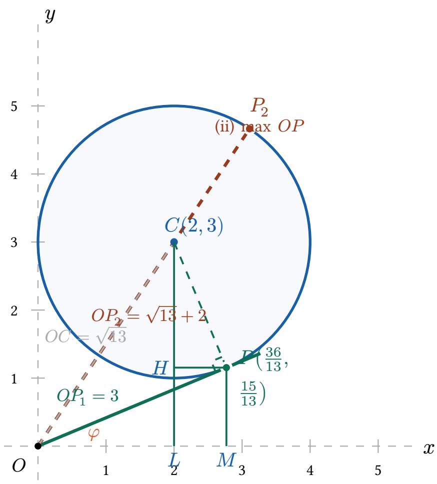
Now $CP = 2, OC = \sqrt{2^2 + 3^2} = \sqrt{13}$

$OP = \sqrt{OC^2 - CP^2} = 3$

From figure $OM = OL + LM = OL + HP \Rightarrow OP\cos\theta = 2 + 2\sin\theta$ or $3\cos\theta = 2 + 2\sin\theta$

$\Rightarrow \cos\theta = \frac{12}{13}, \sin\theta = \frac{5}{13}$

$P = \left(\frac{36}{13}, \frac{15}{13}\right)$




$OP$ will be maximum if $P$ becomes the point where extended part of $OC$ cuts the circle. Let this point be $P_2$.

$OP_2 = OC + r = \sqrt{13} + 2$

Slope is $\frac{3}{2} = \tan\alpha$(let) $\Rightarrow P_2 = \left(2 + \frac{4}{\sqrt{13}}, 3 + \frac{6}{\sqrt{13}}\right)$.

107. Given circle is $x^2 + y^2 - 2ax - 2ay + a^2 = 0$. First we find its point of contact with $x$-axis i.e. $y = 0$.

Putting $y = 0$, $x^2 - 2ax + a^2 = 0 \Rightarrow x = a$. Thus, point of contact is $(a, 0)$.

Then we put $x = 0$ to get $y = a$. Thus, point of contact is $(0, a)$(because we get only one point in both the cases the circle touches the axes.)

108. The given circle is $x^2 + y^2 - 16 = 0$ so its center is $(0,0)$ and radius is 4.

The given points are $(2,3)$ and $(1,2)$. The midpoint is $\left(\frac{3}{2}, \frac{5}{2}\right)$.

The slope of the line joining the points is 1 so the perpendicular slope is $-1$. Hence the chord is the line through $\left(\frac{3}{2}, \frac{5}{2}\right)$ with slope $-1$.

So its equation is $y - \frac{5}{2} = -1\left(x - \frac{3}{2}\right) \Rightarrow x + y - 4 = 0$.

So $d = \frac{|0 + 0 - 4|}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.

The length of the chord is $2\sqrt{r^2 - d^2} = 2\sqrt{16 - 8} = 2\sqrt{8} = 4\sqrt{2}$.

109. The given circle is $x^2 + y^2 - 14x + 4y + 28 = 0 \Rightarrow (x-7)^2 + (y+2)^2 = 25$.

So the center is $(7, -2)$ and radius is 5.

The given line is $x - 7y + 4 = 0$. The perpendicular distance from the center to the line is
$\frac{|7-7(-2)+4|}{\sqrt{1+49}} = \frac{25}{\sqrt{50}} = \frac{5}{\sqrt{2}}$.

The length of the chord is $2\sqrt{r^2 - d^2} = 2\sqrt{25 - \frac{25}{2}} = 5\sqrt{2}$.

The midpoint of the chord is the foot of the perpendicular from the center to the line.

Using formula, midpoint is $\left(7 - 1 \times \frac{25}{50}, -2 + 7 \times \frac{25}{50}\right) \Rightarrow \left(\frac{13}{2}, \frac{3}{2}\right)$.

110. The given circles are $x^2 + y^2 + 3x + 5y + 4 = 0$ and $x^2 + y^2 + 5x + 3y + 4 = 0$.

So we get $-2x + 2y = 0$ or $y = x$.

First circle is $\left(x + \frac{3}{2}\right)^2 + \left(y + \frac{5}{2}\right)^2 = \frac{9}{2}$. So the center is $\left(-\frac{3}{2}, -\frac{5}{2}\right)$ and radius is $\frac{3}{\sqrt{2}}$.

The distance from the center to the line $y = x$ is $\frac{|-\frac{3}{2}+\frac{5}{2}|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$.

The length of the common chord is $2\sqrt{r^2 - d^2} = 2\sqrt{\frac{9}{2} - \frac{1}{2}} = 2\sqrt{4} = 4$.

Hence, the length of the common chord is 4.

111. The given circles are $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$.

So we get $-2x - 1 = 0$ or $x = -\frac{1}{2}$ upon solving.

First circle is $(x + 1)^2 + \left(y + \frac{3}{2}\right)^2 = \frac{9}{4}$. So the center is $\left(-1, -\frac{3}{2}\right)$ and radius is $\frac{3}{2}$.

The distance from the center to the chord $x = -\frac{1}{2}$ is $\frac{|-1+\frac{1}{2}|}{1} = \frac{1}{2}$.

The length of the chord is $2\sqrt{r^2 - d^2} = 2\sqrt{\frac{9}{4} - \frac{1}{4}} = 2\sqrt{2}$.

112. The given circles are $(x - a)^2 + (y - b)^2 = c^2$ and $(x - b)^2 + (y - a)^2 = c^2$. So $(x - a)^2 + (y - b)^2 - (x - b)^2 - (y - a)^2 = 0 \Rightarrow 2(a - b)(y - x) = 0$ so $y = x$.

So the common chord lies on $y = x$.

The center is $(a, b)$ and radius is $c$ of the first circle.

The perpendicular distance from $(a, b)$ to the line $y = x$ is $\frac{|a-b|}{\sqrt{2}}$.

The length of the chord is $2\sqrt{c^2 - d^2}$. So it is $2\sqrt{c^2 - \frac{(a-b)^2}{2}}$.

This simplifies to $\sqrt{4c^2 - 2(a - b)^2}$. Hencem the length of the common chord is $\sqrt{4c^2 - 2(a - b)^2}$.

For the circles to touch, the chord length must be zero. So $4c^2 - 2(a - b)^2 = 0$. Hence, the condition is $2c^2 = (a - b)^2$.

113. The given circles are $x^2 + y^2 + 2hx + a^2 = 0$ and $x^2 + y^2 - 2ky - a^2 = 0$.

Common chord's equaiton is $2hx + a^2 + 2ky + a^2 = 0 \Rightarrow hx + ky + a^2 = 0$.

First circle's center is $(-h, 0)$ and radius squared is $h^2 - a^2$.

The perpendicular distance from the center to the chord is $\frac{|-h^2+a^2|}{\sqrt{h^2+k^2}}$.

So $d^2 = \frac{(h^2-a^2)^2}{h^2+k^2}$.

The length of the chord is $2\sqrt{r^2 - d^2} \Rightarrow 2\sqrt{(h^2 - a^2) - \frac{(h^2-a^2)^2}{h^2+k^2}}$.

$= 2\sqrt{(h^2 - a^2)\left(1 - \frac{h^2-a^2}{h^2+k^2}\right)} = 2\sqrt{\frac{(h^2-a^2)(k^2+a^2)}{h^2+k^2}}$.

114. The given circles are $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + bx + ay + c = 0$.

So chord is $(a - b)x + (b - a)y = 0$ which gives $x = y$.

First circle's center is $\left(-\frac{a}{2}, -\frac{b}{2}\right)$ and radius squared is $\frac{a^2+b^2}{4} - c$.

The perpendicular distance from the center to the line $x - y = 0$ is $\frac{|-\frac{a}{2}+\frac{b}{2}|}{\sqrt{2}} = \frac{|a-b|}{2\sqrt{2}}$.

So $d^2 = \frac{(a-b)^2}{8}$.

The length of the chord is $2\sqrt{r^2 - d^2} = 2\sqrt{\frac{a^2+b^2}{4} - c - \frac{(a-b)^2}{8}} = 2\sqrt{\left(\frac{(a+b)^2}{8}\right) - c}$.

So the length is $\sqrt{\frac{(a+b)^2}{2} - 4c}$.

115. The center of the given circle is origin and radius is $a$. The length of perpendicular from center to tangent is equal to radius. Therefore,

$\frac{r}{\sqrt{p^2+q^2}} = a \Rightarrow r^2 = a^2(p^2 + q^2)$.

116. Center of the given circle is $(3, -5)$ and radius is $\sqrt{3^2 + 5^2 + 66} = 10$.

Length of the perpendicular on the given line from center is $\frac{|4.3+3.5+23|}{5} = 10$, which is equal to the radius of the circle.

Hence, the given circle touches the given line.

117. Center of the given circle is the origin and radius is $a$.

Length of the perpendicular on the given line from center is $\frac{|-a|}{\sqrt{\sin^2\theta + \cos^2\theta}} = a$, which is equaal to radius.

Thus, the given line touches the given circle.

118. Center of the given circle is the origin and radius is $a$.

Length of the perpendicular on the given line from center is $\frac{|-1|}{\sqrt{l^2+m^2}} = a \Rightarrow l^2 + m^2 = a^{-2}$.

Thus, locus of $(l, m)$ is the circle $x^2 + y^2 = a^{-2}$.

119. Given circle has center $(2, 4)$ and radius $\sqrt{2^2 + 4^2 + 5} = 5$.

For the given line to touch the circle length of perpendicular from center to the line must be equal to the radius of the circle. Thus,

$\frac{|3.2-4.4-\lambda|}{5} = 5 \Rightarrow 10 + \lambda = \pm 25 \Rightarrow \lambda = 15, -35$.

120. The given line is $(x - 1)\cos\theta + (y - 1)\sin\theta = 1$.

Expand to get $x\cos\theta + y\sin\theta - \cos\theta - \sin\theta - 1 = 0$.

For a fixed point $(h, k)$ to be the center of a circle touched by all these lines, the perpendicular distance from $(h, k)$ to the line must be constant.

So distance is $(|h\cos\theta + k\sin\theta - \cos\theta - \sin\theta - 1|)$.

This becomes $|(h-1)\cos\theta + (k-1)\sin\theta - 1|$.

For this to be independent of $\theta$, we must have $h - 1 = 0$ and $k - 1 = 0$.

So the center is $(1, 1)$. Now the distance becomes constant equal to 1.

Hence, the radius is 1.

Therefore, the required circle is $(x-1)^2 + (y-1)^2 = 1$.

121. The given line is $3x - 16y = 10$. So the required tangents are of the form $3x - 16y + c = 0$.

For the circle $x^2 + y^2 = 16$, the center is $(0, 0)$ and radius is 4.

The distance from the center to the tangent must be equal to the radius. So $\frac{|c|}{\sqrt{3^2+(-16)^2}} = 4 \Rightarrow |c| = 4\sqrt{265}$. Hence, $c = \pm 4\sqrt{265}$.

Therefore the required tangents are $3x - 16y + 4\sqrt{265} = 0$ and $3x - 16y - 4\sqrt{265} = 0$.

122. The given circle is $x^2 + y^2 - 2x - 4y - 4 = 0$. So the center is $(1, 2)$ and radius is 3.

First consider tangents parallel to $3x - 4y - 1 = 0$. Such lines are of the form $3x - 4y + c = 0$.

The distance from the center to the line equals the radius. So $|3(1) - 4(2) + c\frac{|}{5} = 3$.

$\Rightarrow |-5 + c| = 15 \Rightarrow c = 20$ or $c = -10$.

Hence, the tangents are $3x - 4y + 20 = 0$ and $3x - 4y - 10 = 0$.

Now consider tangents perpendicular to $3x - 4y - 1 = 0$.

Slope of the given line is $\frac{3}{4}$ so perpendicular slope is $-\frac{4}{3}$. So the tangents are of the form $4x + 3y + c = 0$.

Again distance condition gives $|4(1) + 3(2) + c\frac{|}{5} = 3 \Rightarrow c = 5$ or $c = -25$.

Hence, the tangents are $4x + 3y + 5 = 0$ and $4x + 3y - 25 = 0$.

123. The given circle is $x^2 + y^2 - 5x + 5y = 0$.

So the center is $\left(\frac{5}{2}, -\frac{5}{2}\right)$ and radius is $\frac{5}{\sqrt{2}}$.

The given line is $7y - x - 5 = 0$.

The distance from the center to this line is $\frac{|-\frac{5}{2}-\frac{35}{2}-5|}{\sqrt{50}} = \frac{25}{\sqrt{50}} = \frac{5}{\sqrt{2}}$.

So the line touches the circle. Now the other parallel tangent is of the form $7y - x + c = 0$.

Again use the distance condition. So $\frac{| -\frac{5}{2} - \frac{35}{2} + c |}{\sqrt{50}} = \frac{5}{\sqrt{2}}$.

This gives $|c - 20| = 25$. So $c = 45$ or $c = -5$.

Since $c = -5$ gives the given line, the other tangent is $7y - x + 45 = 0$.

124. The given circle is $x^2 + y^2 = 15$ so the center is $(0, 0)$ and radius is sqrt(15).

The given line $4x - y + 6 = 0$ has slope 4. So the required tangents have slope $-\frac{1}{4}$.

Hence, their equations are of the form $x + 4y + c = 0$.

The distance from the center to the tangent must be equal to the radius.
So $\frac{|c|}{\sqrt{1+16}} = \sqrt{15}$. Thus $\frac{|c|}{\sqrt{17}} = \sqrt{15}$.

$\Rightarrow |c| = \sqrt{255}$. Hence, $c = \pm\sqrt{255}$.

Therefore the required tangents are $x + 4y + \sqrt{255} = 0$ and $x + 4y - \sqrt{255} = 0$.

125. The given circle is $x^2 + y^2 - 6x + 4y - 3 = 0$. So the center is $(3, -2)$ and radius is 4.

The given line $y = 2x - 1$ has slope 2. So the required tangents have slope $-\frac{1}{2}$.

Hence their equations are of the form $x + 2y + c = 0$.

The distance from the center to the tangent equals the radius. So $\frac{|3+2(-2)+c|}{\sqrt{5}} = 4$.

So $c = 1 \pm 4\sqrt{5}$.

Hence, the required tangents are $x + 2y + 1 + 4\sqrt{5} = 0$ and $x + 2y + 1 - 4\sqrt{5} = 0$.

126. The given circle is $x^2 + y^2 = 25$ so the center is $(0, 0)$ and radius is 5.

A line making an angle $60°$ with the positive $x$ axis has slope $\tan 60° = \sqrt{3}$.

So the required tangents are of the form $y = \sqrt{3}x + c$.

The distance from the center to the tangent must be equal to the radius. So $|c\frac{|}{\sqrt{1+3}} = 5 \Rightarrow |c| = 10$. Hence, $c = \pm 10$.

Therefore, the required tangents are $y = \sqrt{3}x + 10$ and $y = \sqrt{3}x - 10$.

127. The given pair of lines is $x^2 - y^2 + 2y - 1 = 0$.

Rewrite it as $x^2 - (y - 1)^2 = 0$. So the lines are $x = y - 1$ and $x = 1 - y$.

The family of circles touching both lines has its center on the angle bisectors.

The angle bisectors are $x = 0$ and $y = 1$. First take center $(0, k)$.

The radius is the distance from $(0, k)$ to either line. So $r = \frac{|0-k+1|}{\sqrt{2}}$.

Hence, the circle is $x^2 + (y - k)^2 = \frac{(k-1)^2}{2}$.

This gives one family. Now take center $(h, 1)$. The radius is $\frac{|h|}{\sqrt{2}}$.

Hence, the circle is $(x-h)^2+(y-1)^2=\frac{h^2}{2}$.

128. Let $A$ and $B$ be the centers and $r_1$ and $r_2$ the radii of the given circles respectively. Thus, $A=(1,2), B=(0,4), r_1=\sqrt{5}$, and $r_2=2\sqrt{5}$.

$AB=\sqrt{(1-0)^2+(2-4)^2}=\sqrt{5}$

$r_1+r_2=3\sqrt{5}$ and $|r_1-r_2|=\sqrt{5}$

Thus, $AB=|r_1-r_2|$, hence, the two circles touch each other internally.

129. The centers of the given circles are $A(-a,0)$ and $B(0,-b)$ and radii are $r_1=\sqrt{a^2-c^2}$ and $r_2=\sqrt{b^2-c^2}$ respectively.

The circle will touch internally or externally if $AB=r_1+r_2$ or $AB=|r_1-r_2|$

$AB^2=(r_1\pm r_2)^2 \Rightarrow a^2+b^2=r_1^2+r_2^2\pm 2r_1r_2$

Substituting the values and squaring we get $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$.

130. The centers of the given circles are $A(0,0)$ and $B(2a,0)$ respectively, and radii are $a$ for both.

Distance between centers $AB=2a=r_1+r_2$. Hence, the circles touch each other externally.

Let the equations of the circles touching both the given circles be $(x-\alpha)^2+(y-\beta)^2=a^2$ with center $C=(\alpha,\beta)$ and radius $a$.

$AC=r_1+r_3=2a\Rightarrow \alpha^2+\beta^2=4a^2$, and similarly, $BC=r_2+r_3\Rightarrow (2\alpha-a)^2+\beta^2=4a^2$

$\Rightarrow \alpha=a, \beta=\pm\sqrt{3}a$, and thus, we have our required circles.

131. The given circles are $x^2+y^2+2x+2y+1=0$ and $x^2+y^2-4x-6y-3=0$.

First circle is $(x+1)^2+(y+1)^2=1$. Hence, the center is $(-1,-1)$ and radius is 1.

Second circle is $(x-2)^2+(y-3)^2=16$. Hencem the center is $(2,3)$ and radius is 4.

Now find the distance between the centers. So $d=\sqrt{(2+1)^2+(3+1)^2}=\sqrt{9+16}=5$.

Also $r_1+r_2=1+4=5$. Since $d=r_1+r_2$, the circles touch externally.

132. The given circles have centers $(a,b)$ and $(b,a)$ and both have radius $c$.

The distance between the centers is $\sqrt{(a-b)^2+(b-a)^2}=\sqrt{2(a-b)^2}=\sqrt{2}|a-b|$.

For the circles to touch externally, the distance must be equal to $2c$.

So $\sqrt{2}|a-b|=2c$. Hence $|a-b|=\sqrt{2}c$.

Thus $a-b=\pm\sqrt{2}c$. So the condition is $a=b\pm\sqrt{2}c$.

133. The given circles are $x^2+y^2+2ux+2vy=0$ and $x^2+y^2+2u_1x+2v_1y=0$.

Their centers are $(-u, -v)$ and $(-u_1, -v_1)$. Their radii are $\sqrt{u^2+v^2}$ and $\sqrt{u_1^2+v_1^2}$.

The distance between the centers is $\sqrt{(u-u_1)^2+(v-v_1)^2}$.

For the circles to touch, we must have $\sqrt{(u-u_1)^2+(v-v_1)^2} = \sqrt{u^2+v^2} \pm \sqrt{u_1^2+v_1^2}$.

$\Rightarrow (u-u_1)^2+(v-v_1)^2 = u^2+v^2+u_1^2+v_1^2 \pm 2\sqrt{(u^2+v^2)(u_1^2+v_1^2)}$.

$\Rightarrow u^2+v^2+u_1^2+v_1^2-2(uu_1+vv_1)$ equals the right side.

$\Rightarrow -2(uu_1+vv_1) = \pm 2\sqrt{(u^2+v^2)(u_1^2+v_1^2)}$.

$\Rightarrow uu_1+vv_1 = \mp\sqrt{(u^2+v^2)(u_1^2+v_1^2)}$.

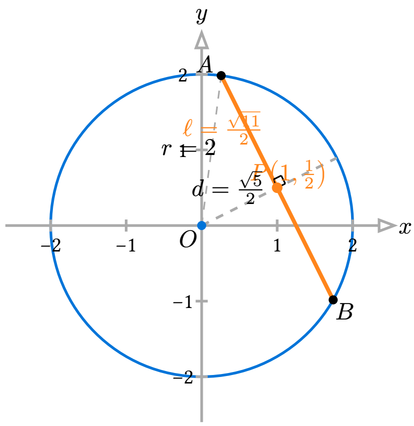
$\Rightarrow (uu_1+vv_1)^2 = (u^2+v^2)(u_1^2+v_1^2)$.

$\Rightarrow u^2u_1^2+v^2v_1^2+2uu_1vv_1 = u^2u_1^2+u^2v_1^2+v^2u_1^2+v^2v_1^2$.

$\Rightarrow 2uu_1vv_1 = u^2v_1^2+v^2u_1^2 \Rightarrow u^2v_1^2-2uu_1vv_1+v^2u_1^2=0$.

So $(uv_1-u_1v)^2=0$. Hence, $uv_1=u_1v$.

134.




Given circle is $x^2+y^2=2^2$ ...(1)

For point $P\left(1, \frac{1}{2}\right)$, $x^2+y^2-4 = 1+\frac{1}{4}-4 < 0$, hence, the point lies inside the circle.

Let $AB$ be any chord of the circle through $P$. Let $OL \perp AB$, then $L$, the will be the middle point of $AB$.

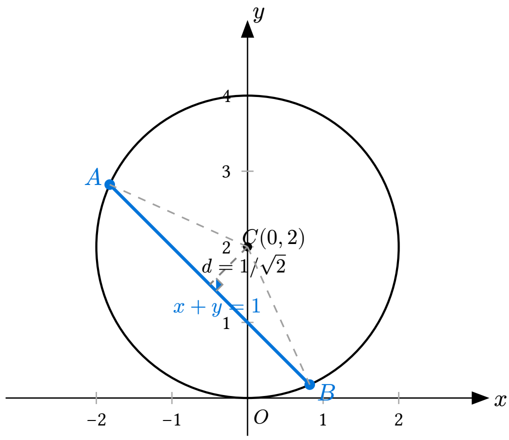
$AB = 2AL = \sqrt{OA^2-OL^2} = 2\sqrt{4-OP^2+LP^2}$

Since $P$ and $O$ are fixed points $OP$ is fixed.

$AB$ will be minimum if $LP$ is minimum and minimum value of $LP$ is 0, when $P$ coincides with $L$.

Thus, minimum value of $AB = 2\sqrt{4-\left(1+\frac{1}{4}\right)} = \sqrt{11}$.

135.




Let $AB$ be the chord whose equation is $x+y-1=0$ ...(1) and given circle is $x^2+y^2-4y=0$ ...(2).

Center of the circle is $(0,2)$. Let $L$ be the mid-point of the chord. Let $\angle ACB = 2\theta$ then $\angle ACL = \angle BCL = \theta$

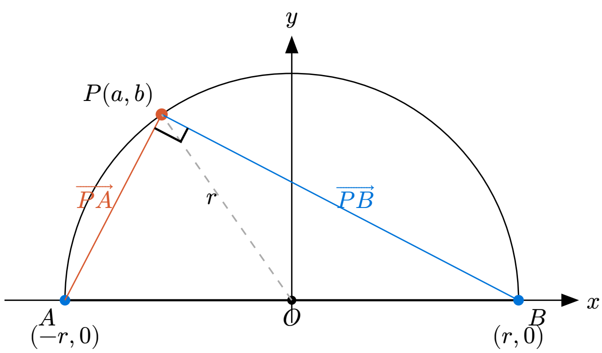
$CL =$ length of perpendicular from $C$ to line (1) $= \frac{|0+2-1|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$AC = 2$(radius of the circle).

From $\triangle\, ACL$, $\cos\theta = \frac{CL}{AC} = \frac{1}{2\sqrt{2}}$

Now angle at circumference $= \frac{1}{2} \times$ angle at the center $= \theta = \cos^{-1}\frac{1}{2\sqrt{2}}$.

136.



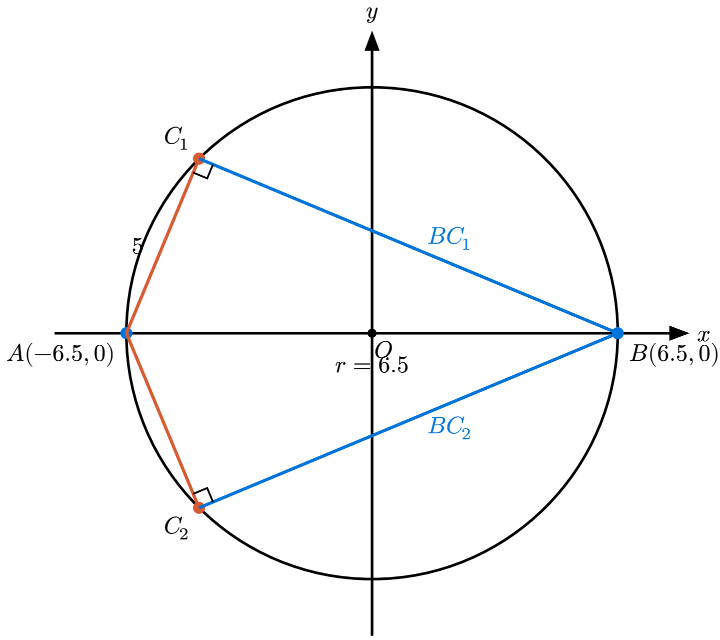

Let $APB$ be a semicircle and $AB$ be a diamtere. Let $O$ be the middle point of $AB$. We take $O$ as the origin and $OB$ as $x$-axis. Let $r$ be the radius of the semicircle. Then $O=(0,0), A=(-r,0)$ and $B=(r,0)$.

Let $P=(x,y)$. Now $OP^2 = r^2 \therefore x^2+y^2=r^2$

and $AP^2 + PB^2 = [(x+a)^2 + y^2] + [(x-a)^2 + y^2] = 2(x^2 + y^2 + r^2) = 4r^2 = (2a)^2 = AB^2$

$\therefore \angle APB = 90^\circ$.

137.




Given $AB = 13$m, $AC = 5$m. Let $OL \perp AC$, then $L$ is the mid-point of $AC$.

$AL = 2.5$m and $AO = 6.5$m. From $\triangle ALO, \cos\theta = \frac{AL}{AO} = \frac{5}{13}$

Slope of $BC_1 = \tan(90^\circ + \theta) = -\cot\theta = -\frac{5}{12}$

Slope of $BC_2 = \tan(-90^\circ - \theta) = \frac{5}{12}$

Equations of $BC_1$ and $BC_2$ are $y = -\frac{5}{12}\left(x - \frac{13}{2}\right)$ and $y = \frac{5}{12}\left(x - \frac{13}{2}\right)$

The join equation is $100x^2 - 576y^2 - 1300x + 4225 = 0$.

138. The given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the internal point is $(\alpha, \beta)$.

Let the center be $(-g, -f)$ and radius be $r$ where $r^2 = g^2 + f^2 - c$.

Let a chord through $(\alpha, \beta)$ be at perpendicular distance $d$ from the center.

The length of the chord is $2\sqrt{r^2 - d^2}$.

For a fixed point inside the circle, the least chord occurs when the chord is perpendicular to the line joining the center and the point.

In that case the distance from the center to the chord is the distance between the center and the point.

So $d^2 = (\alpha + g)^2 + (\beta + f)^2$.

Thus the least length is $2\sqrt{r^2-d^2} \Rightarrow 2\sqrt{g^2+f^2-c-((\alpha+g)^2+(\beta+f)^2)}$.

$= 2\sqrt{-(\alpha^2+\beta^2+2g\alpha+2f\beta+c)}$.

139. Equation of any curve through the point of intersection of given lines and coordinate axes is $(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)+kxy=0$

If this is a circle then coeff. of $x^2$ = coeff. of $y^2 \Rightarrow a_1a_2=b_1b_2$.

140. The given circle is $x^2+y^2=k^2$ so the radius is $k$. The line is $y-x-3=0$.

The perpendicular distance from the center $(0,0)$ to the chord is

$d=\frac{|-3|}{\sqrt{2}}=\frac{3}{\sqrt{2}}$.

Let the chord subtend angle $30^{\circ}$ at a point on the major segment.

Then the angle subtended at the center by the chord is $360^{\circ}-2\times30^{\circ}=300^{\circ}$.

The half angle at the center is $150°$.

So $\cos 150° = \frac{d}{k}$. Thus $-\frac{\sqrt{3}}{2}=\frac{\frac{3}{\sqrt{2}}}{k}$.

So $k=3\sqrt{\frac{2}{3}}=\sqrt{6}$.

141. The tangent to the circle $x^2+y^2=5$ at $(1,-2)$ is

$xx_1+yy_1=r^2$. So $x-2y=5$.

Now consider the second circle $x^2+y^2-8x+6y+20=0$. So the center is $(4,-3)$ and radius is $\sqrt{5}$.

Find the distance from the center to the line $x-2y-5=0$. So $\frac{|4+6-5|}{\sqrt{5}}=\frac{5}{\sqrt{5}}=\sqrt{5}$.

This equals the radius, so the line is tangent.

Put $x=2y+5$ into the circle. So $(2y+5-4)^2+(y+3)^2=5$. This gives $(2y+1)^2+(y+3)^2=5$.

So $5y^2+10y+5=0 \Rightarrow y^2+2y+1=0$. So $(y+1)^2=0$ and $y=-1$. Then $x=3$.

Hence, the point of contact is $(3,-1)$.

142. The given family is $x^2+y^2-2x-2\lambda y-8=0$. Write it as $x^2+y^2-2x-8-2\lambda y=0$.

For fixed points, the equation must hold for all $\lambda$. So $y=0$(for $x$-axis) and $x^2-2x-8=0$.

Thus $x=4$ or $x=-2$. Hence the fixed points are $A(4,0)$ and $B(-2,0)$.

So $2x+2yy'-2-2\lambda y'=0$. Thus $y'=\frac{1-x}{y-\lambda}$.

At $A(4,0)$ we get slope $\frac{3}{\lambda}$. So tangent at $A$ is $y=\left(\frac{3}{\lambda}\right)(x-4)$.

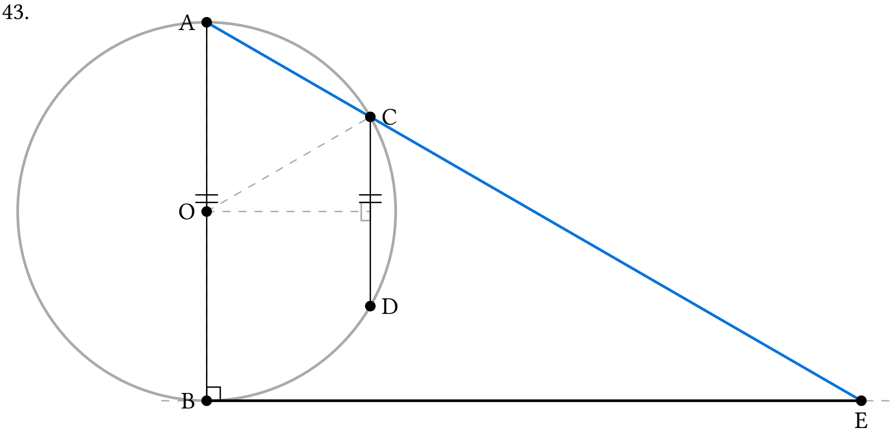
At $B(-2,0)$ we get slope $-\frac{3}{\lambda}$. So tangent at $B$ is $y=\left(-\frac{3}{\lambda}\right)(x+2)$.

Solving gives $x=1$ and $y=-\frac{9}{\lambda}$. This point lies on $x+2y+5=0$.

So $1+2\left(-\frac{9}{\lambda}\right)+5=0$. Thus, $6-\frac{18}{\lambda}=0$ so $\lambda=3$.

Hence the required circle is $x^2+y^2-2x-6y-8=0$.

143.




Let $O$ be the center of the circle which is taken as the origin. Let $a$ be the radius of the circle. Now $A=(0,a), B=(0,-a)$. Since $CD \parallel AB$ and $2CD=AB$

Let $CL \perp CD$. $CL=\frac{CD}{2}=\frac{a}{2}$

In $\triangle OLC, OL=\sqrt{OC^2-CL^2}=\frac{\sqrt{3}a}{2}$

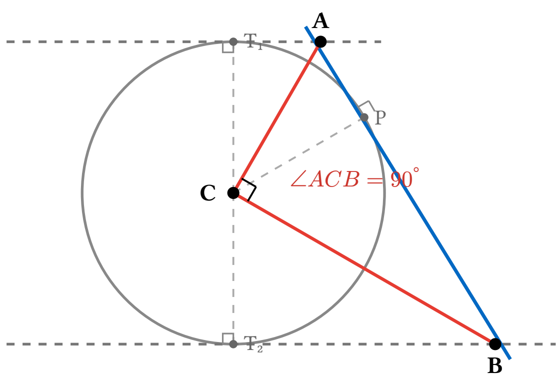
$C=\left(\frac{\sqrt{3}}{2}a, \frac{a}{2}\right), D=\left(\frac{\sqrt{3}}{2}a, -\frac{a}{2}\right)$

Now equation of the circle is $x^2+y^2=a^2$ and equation of tangent at $(0,-a)$ is $-ay=a^2 \Rightarrow y=-a$

Equation of $AC$ is $y-a=\frac{0-\frac{\sqrt{3}}{2}a}{a-\frac{a}{2}}(x-0)$. Solving this with $y=-a$ we get $E=\left(2\sqrt{3}a,-a\right)$

Thus, $AE=2.AB$.

144.



Let the center $C$ of the circle be taken as the origin and let $a$ be the radius of the given circle. Let $CX$ and $CY$ be the $x$ and $y$ axes respectively. Let the two parallel tangents to the circle at $Q$ and $R$ be $y = a$ and $y = -a$.

Equation of the circle is $x^2 + y^2 = a^2$ ...(1) and equation of any other tangent at point $P$ be $y = mx + a\sqrt{1 + m^2}$ ...(2)

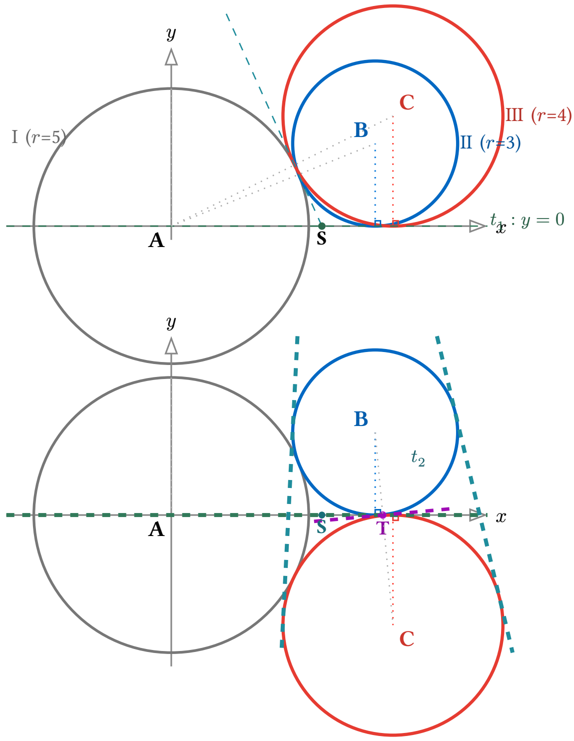
Let $A$ and $B$ be the points of intersection of tangent (2) with the lines $y = a$ and $y = -a$ respectively, then

$A = \left(\frac{a-a\sqrt{1+m^2}}{m}, a\right)$ and $B = \left(\frac{-a-a\sqrt{1+m^2}}{m}, -a\right)$

Slope of $AC = \frac{am}{a\left(1-\sqrt{1+m^2}\right)} = m_1$(let) and slope of $BC = \frac{am}{a+\sqrt{1+m^2}} = m_2$(let)

$m_1 m_2 = -1$, hence, $\angle ACB = 90^{\circ}$.

145.




Given $A$ is the origin, which is the center of circle I, $AX$ and $AY$ are the $x$ and $y$ axes respectively. $B$ and $C$ are the center of the circles II and III respectively, and their radii are 3 and 4 respectively.

Since circles I and II touch each other externally $\therefore AB = 8$ and since circles I and III touch each other externally $\therefore AC = 9$.

Let $BD, CE \perp x$-axis. Then $AD = \sqrt{8^2 - 3^2} = \sqrt{55}$ and $AE = \sqrt{9^2 - 4^2} = \sqrt{65}$.

Since both circles I and II touch $x$-axis, therefore, $y = 0$ is their common tangent. Let $BC$ meet $y = 0$ at $H$. Then one more tangent will pass through $H$ and $H$ will divided $BC$ internally or externally in the ratio $3 : 4$ according as circles II and III lie in different qudrants or in the same quadrant.

**Case I:** When circles II and III lie in the first and fourth quadrant respectively.

In this case $B = \left(\sqrt{55}, 3\right)$ and $C = \left(\sqrt{65}, -4\right)$.

From similar $\triangle BDH$ and $\triangle CEH$, $\frac{DH}{HE} = \frac{BD}{CE} = \frac{3}{4}$

Hence, $H$ divides $DE$ internally in the ratio $3 : 4$. Thus, $H = \left(\frac{3\sqrt{65}+4\sqrt{55}}{7}, 0\right)$

Equation of any line through $H$ will be $y = m\left(x - \frac{3\sqrt{65}+4\sqrt{55}}{7}\right)$

$\Rightarrow 7mx - 7y - m\left(3\sqrt{65} + 4\sqrt{55}\right) = 0$ ...(1)

If (1) is tangent to circle II then

$\frac{\left|7m\sqrt{55} - 7.3 - m\left(3\sqrt{65}+4\sqrt{55}\right)\right|}{7\sqrt{1+m^2}} = 3$

$\Rightarrow m = 0, \frac{126\left(\sqrt{55}-\sqrt{66}\right)}{9\left(71-10\sqrt{143}\right)}$

Thus, we have equation for common tangents.

**Case II:** When both circle II and III lie in the first qudrant.

In this case $B = \left(\sqrt{55}, 3\right)$ and $C = \left(\sqrt{65}, 4\right)$.

One common tangent $y = 0$ meeting $BC$ at $H$ and $H$ will divide $BC$ extrenally in the ratio $3 : 4$.

Thus, $H = \left(4\sqrt{55} - 3\sqrt{65}, 0\right)$.

Now we can proceed like case 1 to find the other common tangent as well as case 3 when both the circles will lie in fourth quadrant.

146. Center of the circle is $(1, 2)$ and the point is $(2, 3)$.

The equation of the normal will be equation of line passing through these points, which is

$y - 3 = \frac{2-3}{1-2}(x - 2) \Rightarrow x - y + 1 = 0.$

147. The equation of the circle is $(x + 4)(x - 6) + (y - 4)(y + 1) = 0 \Rightarrow x^2 + y^2 - 2x - 3y - 28 = 0$

Putting $x = 0$ gives us $y^2 - 3y - 28 = 0 \therefore y = 7, -4$

Hence we take $A$ as $(0, 7)$ and $B$ as $(0, -4)$.

Equation of tangent at $A$ is $x.0 + y.7 - (x + 0) - 3\frac{y+7}{2} - 28 = 0 \Rightarrow 2x - 11y = -77$

Equation of tangent at $B$ is $x.0 + y(-4) - (x + 0) - 3\frac{y-4}{2} - 28 = 0 \Rightarrow 2x + 11t = -44$

Solving the two tangents givens us $Q\left(-\frac{121}{4}, \frac{3}{2}\right)$.

Thus, area of the $\triangle AQB = \frac{363}{8}$.

148. Equation of the tangent is $xx_1 + yy_1 - 2(x + x_1) - 3(y + y_1) - 12 = 0$.

Substituting $(x_1, y_1) = (-1, -1)$ gives $-x - y - 2(x - 1) - 3(y - 1) - 12 = 0$.

Simplify to get $-3x - 4y - 7 = 0$. Hence, the equation of the tangent is $3x + 4y + 7 = 0$.

149. The given circle is $x^2 + y^2 - 7x - 5y + 18 = 0$. We find the tangent at $(4, 3)$.

So $xx_1 + yy_1 - 7\frac{x+x_1}{2} - 5\frac{y+y_1}{2} + 18 = 0$.

Substituting $(4, 3)$ gives $4x + 3y - 7\frac{x+4}{2} - 5\frac{y+3}{2} + 18 = 0$.

Simplifying $x + y - 7 = 0$.

Now find the tangent at $(3, 2)$. So $3x + 2y - 7\frac{x+3}{2} - 5\frac{y+2}{2} + 18 = 0$.

$\Rightarrow x + y - 5 = 0$.

Both tangents have slope $-1$. Hence, they are parallel.

150. The given circle is $x^2 + y^2 = 169$. The tangent at a point $(x_1, y_1)$ on this circle is $xx_1 + yy_1 = 169$.

At $(5, 12)$ the tangent is $5x + 12y = 169$. At $(12, -5)$ the tangent is $12x - 5y = 169$.

From $5x + 12y = 169$ we get slope $-\frac{5}{12}$. From $12x - 5y = 169$ we get slope $\frac{12}{5}$.

Their product is $-1$ so the tangents are perpendicular. Solving $5x + 12y = 169$ and $12x - 5y = 169$ gives $x = 17$ and $y = 7$.

Hence, the point of intersection is $(17, 7)$.

151. The equation of the tangent at $(\alpha, \beta)$ is $x\alpha + y\beta = r^2$.

To find the intercepts, put $y = 0$. Then $x\alpha = r^2$ so $x = \frac{r^2}{\alpha}$.

So point $A$ is $\left(\frac{r^2}{\alpha}, 0\right)$.

Now put $x = 0$. Then $y\beta = r^2$ so $y = \frac{r^2}{\beta}$. So point $B$ is $\left(0, \frac{r^2}{\beta}\right)$.

Now the area of triangle $OAB$ is $\frac{1}{2} \times OA \times OB$.

So area is $\frac{1}{2} \times \frac{r^2}{|\alpha|} \times \frac{r^2}{|\beta|} = \frac{1}{2}\frac{r^4}{|\alpha\beta|}$.

152. The given circle is $x^2 + y^2 - 2x - 4y - 20 = 0 \Rightarrow (x - 1)^2 + (y - 2)^2 = 25$.

So the center is $A(1, 2)$. Tangent at $(x_1, y_1)$ is $xx_1 + yy_1 - (x + x_1) - 2(y + y_1) - 20 = 0$.

Tangent at $(1, 7)$ is $x + 7y - (x + 1) - 2(y + 7) - 20 = 0 \Rightarrow y = 7$.

Tangent at $(4, -2)$ is $4x - 2y - (x + 4) - 2(y - 2) - 20 = 0 \Rightarrow 3x - 4y - 20 = 0$.

Solving the two tangents we get point of intersection as $C(16, 7)$.

Split $ABCD$ into triangles $ABC$ and $ADC$.

Triangle $ABC$ has base $BC = 15$ and height 5. So area is $\frac{1}{2} \times 15 \times 5 = \frac{75}{2}$.

Triangle $ADC$ has base $DC = \sqrt{(16-4)^2 + (7+2)^2} = 15$ and height 5. So area is $\frac{75}{2}$.

Hence, total area is 75.

153. The given circle is $x^2 + y^2 - 2x - 4y + 3 = 0 \Rightarrow (x-1)^2 + (y-2)^2 = 2$.

So the center is $(1, 2)$ and radius is sqrt(2). The given line is $x + y - 5 = 0$.

The distance from the center to the line is $\frac{|1+2-5|}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

This equals the radius, so the line touches the circle.

Put $y = 5 - x$ in the circle. So $(x-1)^2 + (3-x)^2 = 2$.

Expanding gives $x^2 - 2x + 1 + x^2 - 6x + 9 = 2 \Rightarrow 2x^2 - 8x + 10 = 2$.

$2x^2 - 8x + 8 = 0$ so $x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0$ so $x = 2$. Then $y = 3$.

Hence, the point of contact is $(2, 3)$.

154. The tangent to the circle $x^2 + y^2 = 5$ at $(1, -2)$ is

Tangent at $(x_1, y_1)$ is $xx_1 + yy_1 = r^2$. So $x - 2y = 5$.

Now consider the second circle $x^2 + y^2 - 8x + 6y + 20 = (x-4)^2 + (y+3)^2 = 5$.

So the center is $(4, -3)$ and radius is $\sqrt{5}$.

The distance from the center to the line $x - 2y - 5 = 0$. So $\frac{|4+6-5|}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$.

This equals the radius, so the line is tangent.

Put $x = 2y + 5$ into the circle. So $(2y + 5 - 4)^2 + (y+3)^2 = 5 \Rightarrow y^2 + 2y + 1 = 0$.

So $(y+1)^2 = 0$ and $y = -1$. Then $x = 3$.

Hence, the point of contact is $(3, -1)$.

155. The given circles are $x^2 + y^2 - 10x + 4y - 20 = 0$ and $x^2 + y^2 + 14x - 6y + 22 = 0$.

First circle is $(x-5)^2 + (y+2)^2 = 49$ so center is $(5, -2)$ and radius is 7.

Second circle is $(x+7)^2 + (y-3)^2 = 36$ so center is $(-7, 3)$ and radius is 6.

Distance between centers is $\sqrt{(5+7)^2 + (-2-3)^2} = \sqrt{144 + 25} = 13$.

Since $r_1 + r_2 = 7 + 6 = 13$, the circles touch externally.

The point of contact lies on the line joining the centers and divides it in the ratio 7 : 6.

So using section formula $x = \frac{7(-7)+6(5)}{13} = -\frac{19}{13}$ and $y = \frac{7(3)+6(-2)}{13} = \frac{9}{13}$

Hence the point of contact is $\left(-\frac{19}{13}, \frac{9}{13}\right)$.

Tangent at $(x_1, y_1)$ on first circle is $xx_1 + yy_1 - 5(x + x_1) + 2(y + y_1) - 20 = 0 \Rightarrow 19x - 9y + 110 = 0$.

156. The given circle is $x^2 + y^2 = 2$ so the center is $(0, 0)$ and radius is sqrt(2).

The given line is $y - x - 2 = 0$.

The distance from the center to the line is $\frac{|0-0-2|}{\sqrt{2}} = \sqrt{2}$.

This equals the radius, so the line touches the circle.

Put $y = x + 2$ in the circle. So $x^2 + (x + 2)^2 = 2$. Thus, $(x + 1)^2 = 0$ so $x = -1$.

Then $y = 1$. Hence, the point of contact is $(-1, 1)$.

157. The given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Its center is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

For the line $lx + my + n = 0$ to touch the circle, the distance from the center to the line must equal the radius.

So the condition is $\frac{|-lg-mf+n|}{\sqrt{l^2+m^2}} = \sqrt{g^2 + f^2 - c}$.

Squaring both sides gives $(-lg - mf + n)^2 = (l^2 + m^2)(g^2 + f^2 - c)$.

Point of contact is the foot of the perpendicular from the center to the line.

So the coordinates are $x = -g - \frac{l(-lg-mf+n)}{l^2+m^2}$ and $y = -f - \frac{m(-lg-mf+n)}{l^2+m^2}$.

158. The given circle is $x^2 + y^2 = 10x \Rightarrow (x - 5)^2 + y^2 = 25$. So the center is $(5, 0)$ and radius is 5.

The line is $3x + 4y - k = 0$. For tangency, the distance from the center to the line equals the radius.

So $\frac{|15-k|}{5} = 5 \Rightarrow |15 - k| = 25$. So $k = 40$ or $k = -10$.

Now find the point of contact using foot of perpendicular from $(5, 0)$ to the line.

For $k = 40$, $x = 5 - 3\frac{15-40}{25} = 8$ and $y = 0 - 4\frac{15-40}{25} = 4$ So point is $(8, 4)$.

For $k = -10$, $x = 5 - 3\frac{15+10}{25} = 2$ and $y = 0 - 4\frac{15+10}{25} = -4$ So point is $(2, -4)$.

Hence $k = 40$ or $k = -10$ and the points of contact are $(8, 4)$ and $(2, -4)$.

159. The given circle is $x^2 + y^2 = 5$ so the center is $(0, 0)$.

The normal at a point on a circle is the line joining the center to that point.

So the normal passes through $(0, 0)$ and $(1, 2)$. The slope is 2.

Hence, the equation is $y = 2x$.

160. The given circle is $x^2 + y^2 = 2x \Rightarrow (x - 1)^2 + y^2 = 1$ so the center is $(1, 0)$.

The given line $x + 2y = 3$ has slope $-\frac{1}{2}$. So the required normal must also have slope $-\frac{1}{2}$.

The normal to a circle passes through the center.

Hence, the normal is the line through $(1, 0)$ with slope $-\frac{1}{2}$. So its equation is $y = -\frac{1}{2}(x-1)$.

Thus, $x + 2y - 1 = 0$.

161. Given circle is $x^2 + y^2 - 6x - 10y + k = 0$. Let $P = (1, 4)$. Since $P$ lies inside the circle $17 - 6 - 40 + k < 0 \Rightarrow k < 29$.

Let $H$ be the center and $a$ the radius of the circle, then $H = (3, 5)$ and $a = \sqrt{34 - k}$.

Since the circle neither cuts the $x$-axis nor touches it $\therefore a < |5 \Rightarrow k > 9$.

Again since the circle neither cuts the $y$-axis not touches it $\therefore a < |3| \Rightarrow k > 25$

Combining the conditions we have $25 < k < 29$.

162. Lenght of tangent is $\sqrt{5^2 + 1^2 + 6.5 - 4.1 - 3} = 7$.

163. Given circles are $x^2 + y^2 - 2\lambda x - c^2 = 0$ ...(1) where $\lambda$ is a variable.

Let the three values of $\lambda$ be $\lambda_1, \lambda_2$ and $\lambda_3$. Let $A, B$ and $C$ be the centers of the three circles respectively, then

$A = (\lambda_1, 0), B = (\lambda_2, 0)$ and $C = (\lambda_3, 0)$. If $O(0, 0)$ be the origin, then

$OA = |\lambda_1|, OB = |\lambda_2|$ and $OC = |\lambda_3|$.

Given that $|\lambda_1|, |\lambda_2|, |\lambda_3|$ are in G.P. $\therefore |\lambda_2|^2 = |\lambda_1\|\lambda_3|$

Equation of another circle is $x^2 + y^2 = c^2$. Let $P(\alpha, \beta)$ be any circle on this point, then

$\alpha^2 + \beta^2 - c^2 = 0$

Lengths of tangents from $P$ to the three circles are $p_1 = \sqrt{\alpha^2 + \beta^2 - 2\lambda_1\alpha - c^2} = \sqrt{-2\lambda_1\alpha}, p_2 = \sqrt{-2\lambda_2\alpha}$, and $p_3 = \sqrt{-2\lambda_3\alpha}$

$p_1p_3 = \sqrt{4\lambda_1\lambda_3\alpha^2}$. We see that $\lambda_1\lambda_3 > 0 \Rightarrow |\lambda_1\|\lambda_3| = \lambda_1\lambda_3$

Thus, $p_2^2 = p_1p_3$, and hence, p_1, p_2, p_3 are in G.P.

164. Let $P = (\alpha, \beta)$. Given that the lengths of the tangents are equal, therefore,

$\sqrt{\alpha^2 + \beta^2 + \alpha - 3} = \sqrt{\alpha^2 + \beta^2 - \frac{5}{3}\alpha + \beta} = \sqrt{\alpha^2 + \beta^2 + 2\alpha + \frac{7}{4}\beta + \frac{9}{4}}$

Solving we get $\alpha = 0, \beta = -3 \therefore P = (0, -3)$.

Let the equation of the required circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. It passes through $(0, -3)$, therefore,

$-6f + c + 9 = 0$

Equation of the tangent to the circle at $(6, -1)$ is $6x - y + g(x + 6) + f(y - 1) + c = 0$

Given that the equation of the tangent is $x + y - 5 = 0$.

Comparing coefficients we have $g = -\frac{7}{2}, f = \frac{7}{2} \Rightarrow c = 12$

Thus, equation of the circle is $x^2 + y^2 - 7x + 7y + 12 = 0$.

165. The length of tangent from $(f, g)$ to $x^2 + y^2 = 6$ is $\sqrt{f^2 + g^2 - 6}$.

The length of tangent from $(f, g)$ to $x^2 + y^2 + 3x + 3y = 0$ is $\sqrt{f^2 + g^2 + 3f + 3g}$.

Given the first is twice the second. So $\sqrt{f^2 + g^2 - 6} = 2\sqrt{f^2 + g^2 + 3f + 3g}$.

So $f^2 + g^2 - 6 = 4(f^2 + g^2 + 3f + 3g)$.

Simplify to get $0 = 3f^2 + 3g^2 + 12f + 12g + 6$. So $f^2 + g^2 + 4f + 4g + 2 = 0$.

166. The length of the tangent from $(f, g)$ to the circle $x^2 + y^2 = 4$ is $\sqrt{f^2 + g^2 - 4}$.

The second circle is $x^2 + y^2 = 4x$ which is $(x - 2)^2 + y^2 = 4$.

So the length of the tangent from $(f, g)$ to this circle is $\sqrt{(f - 2)^2 + g^2 - 4}$.

Given $\sqrt{f^2 + g^2 - 4} = 4\sqrt{(f - 2)^2 + g^2 - 4}$.

So $f^2 + g^2 - 4 = 16f^2 + 16g^2 - 64f \Rightarrow 15f^2 + 15g^2 - 64f + 4 = 0$.

167. Let $(x_1, y_1)$ be any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

So $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$.

The length of the tangent from $(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c_1 = 0$ is

$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_1}$.

Using the first relation, substitute $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = -c$.

So the length becomes $\sqrt{-c + c_1}$.

168. Let the required point be $(x, y)$. The length of the tangent from $(x, y)$ to $x^2 + y^2 = 1$ is

$\sqrt{x^2 + y^2 - 1}$. For the circle $x^2 + y^2 - 8x + 15 = 0$ it is $\sqrt{x^2 + y^2 - 8x + 15}$.

For the circle $x^2 + y^2 + 10y + 24 = 0$ it is $\sqrt{x^2 + y^2 + 10y + 24}$.

Given $x^2 + y^2 - 1 = x^2 + y^2 - 8x + 15 \Rightarrow x = 2$.

Again $x^2 + y^2 - 1 = x^2 + y^2 + 10y + 24 \Rightarrow y = -\frac{5}{2}$.

Hence, the required point is $\left(2, -\frac{5}{2}\right)$.

169. Let the required point be $(x, y)$. The length of the tangent from $(x, y)$ to $x^2 + y^2 - 4x + 7 = 0$ is

$\sqrt{x^2 + y^2 - 4x + 7}$. For the circle $2x^2 + 2y^2 - 3x + 5y + 9 = 0$ divide by 2.

So it becomes $x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y + \frac{9}{2} = 0$.

Hence, the length is $\sqrt{x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y + \frac{9}{2}}$.

For the circle $x^2 + y^2 + y = 0$ the length is $\sqrt{x^2 + y^2 + y}$.

Equating first and second. So $x^2 + y^2 - 4x + 7 = x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y + \frac{9}{2}$.

Thus, $-5x - 5y + 5 = 0$ so $x + y = 1$.

Now equate first and third. So $x^2 + y^2 - 4x + 7 = x^2 + y^2 + y$. So $y = -4x + 7$.

Solve with $x + y = 1$. So $x - 4x + 7 = 1$.

$\Rightarrow x = 2$ and $y = -1$. Hence, the point is $(2, -1)$.

So length is $\sqrt{4 + 1 - 8 + 7} = \sqrt{4} = 2$.

170. Let the point be $(x, y)$. For the circle $x^2 + y^2 + 2g_i x + 5 = 0$, the length of the tangent from $(x, y)$ is $t_i^2 = x^2 + y^2 + 2g_i x + 5$.

Now consider $(g_2 - g_3)t_1^2 + (g_3 - g_1)t_2^2 + (g_1 - g_2)t_3^2$.

Substitute $t_i^2$. So it becomes $(g_2 - g_3)(x^2 + y^2 + 2g_1 x + 5) + (g_3 - g_1)(x^2 + y^2 + 2g_2 x + 5) + (g_1 - g_2)(x^2 + y^2 + 2g_3 x + 5)$.

The coefficient of $(x^2 + y^2 + 5)$ is $(g_2 - g_3) + (g_3 - g_1) + (g_1 - g_2) = 0$.

Now consider the remaining terms.

So we get $2x[(g_2 - g_3)g_1 + (g_3 - g_1)g_2 + (g_1 - g_2)g_3]$.

Expanding inside. So $g_1 g_2 - g_1 g_3 + g_2 g_3 - g_1 g_2 + g_1 g_3 - g_2 g_3 = 0$.

Hence, the whole expression is 0.

171. Let the point be $(x, y)$. The length of the tangent from $(x, y)$ to $x^2 + y^2 = a^2$ is $\sqrt{x^2 + y^2 - a^2}$.

The second circle is $(x - a)^2 + y^2 = a^2$. So the length of the tangent from $(x, y)$ to this circle is $\sqrt{(x - a)^2 + y^2 - a^2}$.

Given the first is four times the second. So $\sqrt{x^2 + y^2 - a^2} = 4\sqrt{(x - a)^2 + y^2 - a^2}$.

$\Rightarrow x^2 + y^2 - a^2 = 16((x - a)^2 + y^2 - a^2) \Rightarrow x^2 + y^2 - a^2 = 16(x^2 + y^2 - 2ax)$.

So $0 = 15x^2 + 15y^2 - 32ax + a^2$. Hence, the point lies on the required circle.

172. The equation of the pair of tangents from $(0, 1)$ is given by $T^2 = SS_1$.

Here $S = x^2 + y^2 - 2x + 4y$ and $S_1 = 0^2 + 1^2 - 2(0) + 4(1) = 5$.

Now $T$ is $x \times 0 + y \times 1 - (x + 0) + 2(y + 1)$. So $T = y - x + 2y + 2 = -x + 3y + 2$.

Thus the equation is $(-x + 3y + 2)^2 = 5(x^2 + y^2 - 2x + 4y)$

$\Rightarrow 3x^2 - 2y^2 + 3xy - 3x + 4y - 2 = 0$.

173. Let $(x_1, y_1)$ be any point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. So $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$.

The second circle is $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$.

The length of the tangent from $(x_1, y_1)$ to the second circle is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha}$.

Using the first relation, substitute $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = -c$.

So the length becomes $\sqrt{-c + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha}$.

This simplifies to $\sqrt{(g^2 + f^2 - c)\cos^2\alpha}$. So the tangent length is $\sqrt{g^2 + f^2 - c}\cos\alpha$.

The radius of the second circle is $\sqrt{g^2 + f^2 - (c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha)}$.

$= \sqrt{(g^2 + f^2 - c)\sin^2\alpha}$.

Let $\theta$ be the angle between the tangents. Then $\tan\left(\frac{\theta}{2}\right) = \frac{r}{d}$ where $d$ is the tangent length.

So $\tan\left(\frac{\theta}{2}\right) = \tan\alpha$. Hence, $\theta = 2\alpha$.

174. The given circle is $x^2 + y^2 = 25$ so the center is $(0, 0)$ and radius is 5.

Let the tangent from $(1, -7)$ have slope $m$. So its equation is $y + 7 = m(x - 1)$.

This gives $mx - y - m - 7 = 0$.

For tangency, the distance from the center to the line equals the radius. So $\frac{|-m-7|}{\sqrt{m^2+1}} = 5$.

$\Rightarrow m = \frac{4}{3}$ or $m = -\frac{3}{4}$.

Hence, the tangents are $y + 7 = \frac{4}{3}(x - 1)$ and $y + 7 = -\frac{3}{4}(x - 1)$. Their slopes are $\frac{4}{3}$ and $-\frac{3}{4}$.

Their product is $-1$ so they are perpendicular.

175. The given circle is $x^2 + y^2 = 16$. From the point $(9, 0)$ the length of the tangent is $\sqrt{9^2 - 16} = \sqrt{65}$ which is real. Hence, two tangents can be drawn.

The equation of the pair of tangents is given by $T^2 = SS_1$.

Here $S = x^2 + y^2 - 16$ and $S_1 = 81 - 16 = 65$.

Now $T = xx_1 + yy_1 - 16 = 9x - 16$.

So the equation is $(9x - 16)^2 = 65(x^2 + y^2 - 16)$.

Thus, $16x^2 - 65y^2 - 288x + 1296 = 0$.

So $\tan\theta = 2\frac{\sqrt{0-(16)(-65)}}{16-65} = -8\frac{\sqrt{65}}{49}$.

176. The given circle is $x^2 + y^2 = 25$ so the center is $(0, 0)$ and radius is 5.

Let the tangent through $(7, 1)$ have slope $m$.

So its equation is $y - 1 = m(x - 7)$. This gives $mx - y - 7m + 1 = 0$.

For tangency, the distance from the center to the line equals the radius. So $\frac{|-7m+1|}{\sqrt{m^2+1}} = 5$. Solve to get $m = \frac{4}{3}$ or $m = -\frac{3}{4}$.

Hence the tangents are $y - 1 = \frac{4}{3}(x - 7)$ and $y - 1 = -\frac{3}{4}(x - 7)$.

177. The given circle is $x^2 + y^2 + 2gx + 2fy + k^2 = 0$.

The equation of the pair of tangents from the origin is given by $T^2 = SS_1$.

Here $S = x^2 + y^2 + 2gx + 2fy + k^2$ and $S_1 = k^2$. Now $T = gx + fy + k^2$.

So the equation is $(gx + fy + k^2)^2 = k^2(x^2 + y^2 + 2gx + 2fy + k^2)$.

This is the required pair of tangents.

Now find the intercept on the line $y = h$. Substitute $y = h$.

So $(gx + fh + k^2)^2 = k^2(x^2 + h^2 + 2gx + 2fh + k^2)$.

The intercept is the distance between the two roots.

So length is $2\frac{\sqrt{(g^2-k^2)(h^2+k^2+2fh)}}{|k^2-g^2|}$.

Using the relation $g^2 + f^2 - k^2 = r^2$ simplify the expression.

This reduces to $\frac{2hkr}{k^2-g^2}$.

Hence, the intercept is $\frac{2hk}{k^2-g^2}$ times the radius.

178. The given circle is $x^2 + y^2 + 6x + 8y - 11 = 0$. Let the midpoint of the chord be $(1, -1)$.

The chord whose midpoint is $(x_1, y_1)$ is given by $T = S_1$.

Here $S = x^2 + y^2 + 6x + 8y - 11$ and $S_1 = 1^2 + (-1)^2 + 6(1) + 8(-1) - 11 = -11$.

Now $T = xx_1 + yy_1 + 3(x + x_1) + 4(y + y_1) - 11$.

Substitute $(1, -1)$. So $T = x - y + 3(x + 1) + 4(y - 1) - 11$.

Thus, the chord is $T = S_1$ so $4x + 3y - 12 = -11$.

Hence, the equation is $4x + 3y - 1 = 0$.

179. The given circle is $x^2 + y^2 + 6x + 8y + 9 = 0$. Let the midpoint be $(-2, -3)$.

The chord whose midpoint is $(x_1, y_1)$ is given by $T = S_1$.

Here $S = x^2 + y^2 + 6x + 8y + 9$ and $S_1 = (-2)^2 + (-3)^2 + 6(-2) + 8(-3) + 9 = -14$.

Now $T = xx_1 + yy_1 + 3(x + x_1) + 4(y + y_1) + 9$.

Substitute $(x_1, y_1) = (-2, -3)$. So $T = -2x - 3y + 3(x - 2) + 4(y - 3) + 9$.

Simplify to get $x + y - 9$. Thus, the chord is $T = S_1$ so $x + y - 9 = -14$.

Hence, the equation is $x + y + 5 = 0$.

180. The given circle is $x^2 + y^2 + 4x - 2y - 3 = 0$. So the center is $(-2, 1)$.

The given line is $y = x + 2$ or $x - y + 2 = 0$.

The midpoint of the chord is the foot of the perpendicular from the center to the line.

Using the formula for foot of perpendicular from $(x_1, y_1)$ to $ax + by + c = 0$,

$x = x_1 - \frac{a(ax_1+by_1+c)}{a^2+b^2}$ and $y = y_1 - \frac{b(ax_1+by_1+c)}{a^2+b^2}$.

Here $(x_1, y_1) = (-2, 1)$ and $a = 1$, $b = -1$, $c = 2$.

$ax_1 + by_1 + c = -2 - 1 + 2 = -1$.

So $x = -2 - 1\frac{-1}{2} = -\frac{3}{2}$ and $y = 1 - (-1)\frac{-1}{2} = \frac{1}{2}$.

Hence, the midpoint is $\left(-\frac{3}{2}, \frac{1}{2}\right)$.

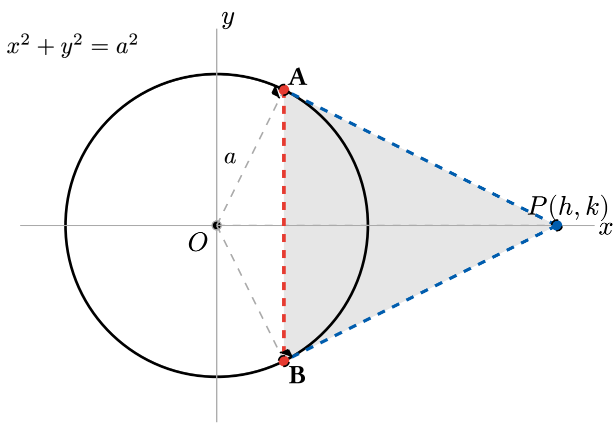
181. The given circle is $x^2 + y^2 - 2x + 4y + 7 = 0$. The chord of contact from a point $(x_1, y_1)$ is given by $T = 0$.

Here $T = xx_1 + yy_1 - (x + x_1) + 2(y + y_1) + 7$.

Substitute $(x_1, y_1) = (1, 2)$. So $T = x + 2y - (x + 1) + 2(y + 2) + 7$.

$\Rightarrow 4y + 10 = 0 \Rightarrow 2y + 5 = 0$.

182.




Given circle is $x^2 + y^2 = a^2$ ...(1)

The equation of the chord of contact $AB$ of tangents drawn from $P(h, k)$ to the circle (1) is $xh + yk = a^2$.

We have to find the area of $\triangle PAB$. From $P(h, k)$ draw $PL \perp AB$. Now

$PL = \frac{h^2+k^2-a^2}{\sqrt{h^2+k^2}}$

Also, $PA = \sqrt{h^2 + k^2 - a^2}$

$AL^2 = AP^2 - PL^2 = \frac{a\sqrt{h^2+k^2-a^2}}{\sqrt{h^2+k^2}}$

$\Delta_{PAB} = \frac{1}{2}AB.PL = AL.PL = \frac{a(h^2+k^2-a^2)^{\frac{3}{2}}}{h^2+k^2}$.

183. Given circles are $x^2 + y^2 = a^2$ ...(1), $x^2 + y^2 = b^2$ ...(2), and $x^2 + y^2 = c^2$ ...(3).

Let $P(\alpha, \beta)$ be any point on (1), then $\alpha^2 + \beta^2 = a^2$ ...(4)

Equation of the chord of contact of the tangents from $P(\alpha, \beta)$ to (2) is

$x\alpha + y\beta - b^2 = 0.$

This chord of contact is tangent to (3), therefore,

$\frac{|0.\alpha+0.\beta-b^2|}{\sqrt{\alpha^2+\beta^2}} = c \Rightarrow b^2 = ac$, and hence, $a, b, c$ are in G.P.

184. Common chord of the circles is $5x - 3y - 10 = 0$. Let this meet the circles at $A$ and $B$. Let the tangents to first circle at $A$ and $B$ meet at $P(\alpha, \beta)$, then $AB$ will be the chord of contact of the tangents to the circle from $P$, therefore, equation of $AB$ will be

$x\alpha + y\beta - 12 = 0$

The two obtained equations are same. Comparing coefficients we have $\frac{\alpha}{5} = \frac{\beta}{-}3 = -\frac{12}{10}$, which yields

$\alpha = 6, \beta - = -\frac{18}{5}.$

185. The given circle is $x^2 + y^2 + 2x - 3 = 0$. The chord of contact from a point $(x_1, y_1)$ is given by $T = 0$.

Here $T = xx_1 + yy_1 + (x + x_1) - 3$. Substitute $(x_1, y_1) = (-3, 2)$.

So $T = -3x + 2y + (x - 3) - 3$.

Hence, the chord of contact is $x - y + 3 = 0$.

186. The given circle is $x^2 + y^2 = 25$. The chord of contact from a point $(x_1, y_1)$ is given by $xx_1 + yy_1 = 25$.

Substitute $(x_1, y_1) = (5, 3)$. So the equation becomes $5x + 3y = 25$.

187. The given circle is $x^2 + y^2 - 4x + 3y - 1 = 0$. The line is $2x + y + 12 = 0$.

Let the points of intersection be $P$ and $Q$. The intersection of tangents at $P$ and $Q$ is given by the pole of the line.

So we find the pole of $2x + y + 12 = 0$ with respect to the circle.

For the circle, the pole of $lx + my + n = 0$ is $\frac{-2gl-mn,-2fm-ln}{l^2+m^2}$.

Here $g = -2$ and $f = \frac{3}{2}$. $\Rightarrow x = \frac{-2(-2)(2)-(1)(12)}{5} = -\frac{4}{5}$ and $y = \frac{-2(\frac{3}{2})(1)-(2)(12)}{5} = -\frac{27}{5}$.

188. Let the given circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Let the two points be $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Since they are conjugate with respect to the circle, we have $x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$.

The lengths of tangents are $t_1^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ and $t_2^2 = x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c$.

$t_1^2 + t_2^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2g(x_1 + x_2) + 2f(y_1 + y_2) + 2c.$

Now consider the square of the distance between the points.

So $P_Q^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2(x_1x_2 + y_1y_2).$

Using the conjugate condition, $x_1x_2 + y_1y_2 = -g(x_1 + x_2) - f(y_1 + y_2) - c$.

$\therefore P_Q^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2g(x_1 + x_2) + 2f(y_1 + y_2) + 2c$.

Thus $P_Q^2 = t_1^2 + t_2^2$. Hence, $P_Q = \sqrt{t_1^2 + t_2^2}$.

189. The given circle is $x^2 + y^2 = 25$. Let the point be $P(4, 6)$.

The chord of contact from $P$ is $4x + 6y = 25$.

The area of the triangle formed by the two tangents and their chord of contact is $\Delta = \frac{r^2 \times \sqrt{S_1}}{\text{distance from center to chord}}$.

Here $S_1 = 4^2 + 6^2 - 25 = 27$. So $\sqrt{S_1} = 3\sqrt{3}$.

The distance from the center $(0, 0)$ to the chord $4x + 6y - 25 = 0$ is

$= 25 \times 3\sqrt{3} \times 2\frac{\sqrt{13}}{25} = 6\sqrt{39}$.

190. The given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the points of contact of tangents from the origin.

The tangent at $(x_1, y_1)$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Since this tangent passes through the origin, substitute $(0, 0)$. So $gx_1 + fy_1 + c = 0$.

Also since $(x_1, y_1)$ lies on the circle, $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$.

Using $gx_1 + fy_1 = -c$, we get $x_1^2 + y_1^2 - c = 0 \Rightarrow x_1^2 + y_1^2 = c$. Similarly $x_2^2 + y_2^2 = c$.

Now consider the circle $x^2 + y^2 + gx + fy = 0$. Substitute $(0, 0)$ and it satisfies the equation.

Now substitute $(x_1, y_1)$. Using $x_1^2 + y_1^2 = c$ and $gx_1 + fy_1 = -c$, we get $c - c = 0$.

So $(x_1, y_1)$ lies on it. Similarly $(x_2, y_2)$ lies on it. Hence this circle passes through $O, P, Q$.

Therefore it is the circumcircle of $\triangle OPQ$.

191. The given circle is $x^2 + y^2 = 25$. Let the tangent through $(7, 1)$ have slope $m$.

So its equation is $y - 1 = m(x - 7)$. This gives $mx - y - 7m + 1 = 0$.

For tangency, the distance from the center $(0, 0)$ to the line equals the radius 5. So $\frac{|-7m+1|}{\sqrt{m^2+1}} = 5$. Solve to get $m = \frac{4}{3}$ or $m = -\frac{3}{4}$.

Hence the tangents are $y - 1 = \frac{4}{3}(x - 7)$ and $y - 1 = -\frac{3}{4}(x - 7)$.

For $m = \frac{4}{3}$, the tangent is $4x - 3y - 25 = 0$.

Solve with the circle. So $x^2 + \frac{(4x-25)^2}{9} = 25$. This gives $x = 4$ and $y = 3$.

For $m = -\frac{3}{4}$, the tangent is $3x + 4y - 25 = 0$. Solve with the circle. So $x^2 + \frac{(25-3x)^2}{16} = 25$.

This gives $x = 3$ and $y = -4$. Hence the points of contact are $(4, 3)$ and $(3, -4)$.

192. Equation of the polar is $x.2 + y.(-1) - 3\left(\frac{x+2}{2}\right) + 4\frac{y-1}{2} - 8 = 0 \Rightarrow x + 2y - 26 = 0$.

193. Let $P(\alpha, \beta)$ be the pole of the given line w.r.t. the given circle. Equation of polar is

$(\alpha + 2)x + (\beta + 3)y + 2\alpha + 3\beta + 9 = 0$

Comparing with the given line $\frac{\alpha+2}{3} = \frac{\beta+3}{5} = \frac{2\alpha+3\beta+9}{17}$

$\Rightarrow \alpha = 1, \beta = 2$. So the required pole is $(1, 2)$.

194. Given circles are $x^2 + y^2 + 6y + 5 = 0 \quad \text{dots}(1)$ and $x^2 + y^2 + 2x + 8y + 5 = 0 \quad$ ...(2). Let $P = (1, -2)$.

Polar of the point $(1, -2)$ w.r.t circle (1) is given by $x + y.(-2) + 3(y - 2) + 5 = 0 \Rightarrow x + y - 1 = 0 \quad$ ...(3)

Polar of this point w.r.t circle (2) is given by $x + y.(-2) + x + 1 + 4(y - 2) + 5 = 0 \Rightarrow x + y - 1 = 0$.

Thus, polars are same. Let $Q(\alpha, \beta)$ be another point for which the polars are same. The polars of this point w.r.t. given circles are

$x\alpha + y\beta + 3(y + \beta) + 5 = 0$ and $x\alpha + y\beta + (x + \alpha) + 4(y + \beta) + 5 = 0$

These two lines are same. Thus, comparing coefficients gives us
$\frac{\alpha+1}{\alpha} = \frac{\beta+4}{\beta+3} = \frac{\alpha+4\beta+5}{3\beta+5}$

Solvig this gives us two points one of whihc is the given point and another point is $(2, -1)$.

195. Let the circle be $x^2 + y^2 = a^2$ and points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Polars of $A$ and $B$ will be $xx_1 + yy_1 - a^2 = 0$ and $xx_2 + yy_2 - a^2 = 0$.

$\frac{AM}{BN} = \frac{|x_1x_2+y_1y_2-a^2|}{\sqrt{x_2^2+y_2^2}} = \frac{|x_2x_2+y_2y_2-a^2|}{\sqrt{x_1^2+y_2^2}} = \frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x_2^2+y_2^2}} = \frac{CA}{CB}$.

196. First find the point of intersection of the lines $4x - y = 11$ and $x - 2y = 1$.

From the second equation we get $x = 1 + 2y$. Substitute in the first equation.

So $4(1 + 2y) - y = 11$. Thus, $x = 3$. So the point is $(3, 1)$.

Now find the polar with respect to $x^2 + y^2 = 7$.

The polar of $(x_1, y_1)$ is $xx_1 + yy_1 = 7$.

So the required equation is $3x + y = 7$.

197. The given circle is $2x^2 + 2y^2 = 11 \Rightarrow x^2 + y^2 = \frac{11}{2}$.

So the polar is $4x - y = \frac{11}{2}$.

198. The given circle is $x^2 + y^2 - 8x + 6y + 4 = 0$. The polar of a point $(x_1, y_1)$ is given by $T = 0$.

So $T = xx_1 + yy_1 - 4(x + x_1) + 3(y + y_1) + 4$.

Substitute $(x_1, y_1) = (1, -5)$. So $T = x - 5y - 4(x+1) + 3(y-5) + 4$.

Hence, the polar is $3x + 2y + 15 = 0$.

199. The polar of $(p, q)$ with respect to $x^2 + y^2 = a^2$ is $px + qy = a^2$.

For this line to touch the circle $(x-c)^2 + (y-d)^2 = b^2$, the distance from its center $(c, d)$ to the line must equal the radius $b$.

So $\frac{|pc+qd-a^2|}{\sqrt{p^2+q^2}} = b \Rightarrow \left(pc + qd - a^2\right)^2 = b^2(p^2 + q^2)$

$\Rightarrow b^2(p^2 + q^2) = \left(a^2 - cp - dq\right)^2$.

200. The given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

The polar of the origin is obtained by putting $(0, 0)$ in $T$.

So the polar is $gx + fy + c = 0$.

For this line to touch the circle $x^2 + y^2 = a^2$, the distance from the center $(0, 0)$ to the line must equal the radius $a$.

So $\frac{|c|}{\sqrt{g^2+f^2}} = a$. Hence, $c^2 = a^2(g^2 + f^2)$.

201. Let the given line be $lx + my + n = 0$. The pole of this line with respect to $x^2 + y^2 = c^2$ is $\left(-c^2 \frac{l}{n}, -c^2 \frac{m}{n}\right)$.

Given this point lies on $x^2 + y^2 = 9c^2$. So $\frac{c^4 l^2}{n^2} + \frac{c^4 m^2}{n^2} = 9c^2$.

Thus, $c^4(l^2 + m^2) = 9c^2 n^2$. So $n^2 = \left(\frac{c^2}{9}\right)(l^2 + m^2)$.

Now consider the circle $9x^2 + 9y^2 = c^2$. Its center is $(0, 0)$ and radius is $\frac{c}{3}$.

The distance from the center to the line is $\frac{|n|}{\sqrt{l^2+m^2}}$.

Using the relation above, this becomes $\frac{c}{3}$.

So the distance equals the radius. Hence the line is tangent to the circle $9x^2 + 9y^2 = c^2$.

202. The given circle is $2x^2 + 2y^2 - 3x + 5y - 7 = 0 \Rightarrow x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y - \frac{7}{2} = 0$. Thus, $g = -\frac{3}{4}$ and $f = \frac{5}{4}$.

The pole $(x_1, y_1)$ of the line $9x + y - 28 = 0$ satisfies that this line is the polar of $(x_1, y_1)$.

So write $T = 0$. Thus, $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

So coefficient of $x$ is $x_1 + g$ and coefficient of $y$ is $y_1 + f$.

Hence, the equation becomes $(x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0$.

Compare with $9x + y - 28 = 0$. So $x_1 + g = 9$ and $y_1 + f = 1$.

Substitute values. So $x_1 - \frac{3}{4} = 9$ hence $x_1 = \frac{39}{4}$. And $y_1 + \frac{5}{4} = 1$ hence $y_1 = -\frac{1}{4}$.

So the pole is $\left(\frac{39}{4}, -\frac{1}{4}\right)$. So it becomes $\left(\frac{9}{4}\right)x + \left(\frac{1}{4}\right)y - 7 = 0$.

Now compare again. So $x_1 + g = \frac{9}{4}$ gives $x_1 = 3$. And $y_1 + f = \frac{1}{4}$ gives $y_1 = -1$.

203. The given circle is $x^2 + y^2 - 7x + 5y - 1 = 0$. Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$.

So $g = -\frac{7}{2}$, $f = \frac{5}{2}$, $c = -1$. Let the pole be $(x_1, y_1)$. The polar of $(x_1, y_1)$ is

$(x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0$. This must represent the line $2x - y + 10 = 0$.

So equate coefficients with a factor $k$: $x_1 + g = 2k$, $y_1 + f = -k$ and $gx_1 + fy_1 + c = 10k$.

Substitute $g = -\frac{7}{2}$, $f = \frac{5}{2}$, $c = -1$.

So $x_1 - \frac{7}{2} = 2k$ hence $x_1 = 2k + \frac{7}{2}$ and $y_1 + \frac{5}{2} = -k$ hence $y_1 = -k - \frac{5}{2}$.

Substitute into the third equation: $\left(-\frac{7}{2}\right)\left(2k + \frac{7}{2}\right) + \left(\frac{5}{2}\right)\left(-k - \frac{5}{2}\right) - 1 = 10k$.

Expand: $-7k - \frac{49}{4} - 5\frac{k}{2} - \frac{25}{4} - 1 = 10k$. $\Rightarrow -7k - 5\frac{k}{2} - \frac{37}{2} - 1 = 10k$.

Write $-1 = -\frac{2}{2}$: $-7k - 5\frac{k}{2} - \frac{39}{2} = 10k \Rightarrow k = -1$.

Now $x_1 = 2k + \frac{7}{2} = -2 + \frac{7}{2} = \frac{3}{2}$ and $y_1 = -k - \frac{5}{2} = 1 - \frac{5}{2} = -\frac{3}{2}$.

204. The given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Let the pole of the line $ax + by + c = 0$ be $(x_1, y_1)$. The polar of $(x_1, y_1)$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 \Rightarrow (x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0$.

This must represent the same line as $ax + by + c = 0$.

So coefficients are proportional. Let the factor be $k$.

$x_1 + g = ak$ and $y_1 + f = bk$ => $gx_1 + fy_1 + c = ck$.

So $x_1 = ak - g$ and $y_1 = bk - f$.

Substitute into the third equation: $g(ak - g) + f(bk - f) + c = ck$.

So $agk - g^2 + bfk - f^2 + c = ck$. Thus, $k(ag + bf - c) = g^2 + f^2 - c$.

Hence, $k = \frac{g^2+f^2-c}{ag+bf-c}$.

So $x_1 = \frac{a(g^2+f^2-c)}{ag+bf-c} - g$ and $y_1 = \frac{b(g^2+f^2-c)}{ag+bf-c} - f$.

205. The given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Its center is $(-g, -f)$.

Let the point be $P(x_1, y_1)$. The polar of $P$ is

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

So $(x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0$.

Thus, the normal vector to the polar is $(x_1 + g, y_1 + f)$.

Now consider the line joining the center $C(-g, -f)$ and the point $P(x_1, y_1)$.

Its direction vector is $(x_1 + g, y_1 + f)$.

Hence, the polar is perpendicular to the line joining the point and the center.

206. The family of circles is $x^2 + y^2 + 2px + c = 0$. Let the given point be $(x_1, y_1)$.

The polar of $(x_1, y_1)$ with respect to the circle is $xx_1 + yy_1 + p(x + x_1) + c = 0$.

So $(x_1 + p)x + y_1y + px_1 + c = 0 \Rightarrow x_1x + y_1y + c + p(x + x_1) = 0$.

Now observe that if $x + x_1 = 0$, the term containing $p$ vanishes.

So the equation reduces to $x_1x + y_1y + c = 0$. Substitute $x = -x_1$.

Then $-x_1^2 + y_1y + c = 0$. So $y = \frac{x_1^2 - c}{y_1}$.

Thus the point $\left(-x_1, \frac{x_1^2 - c}{y_1}\right)$ satisfies the polar for all values of $p$.

Hence, all polars pass through this fixed point.

207. The polar of $(\alpha, \beta)$ with respect to $x^2 + y^2 = a^2$ is $\alpha x + \beta y = a^2$.

For this line to touch the circle $(x - a)^2 + y^2 = a^2$, the distance from its center $(a, 0)$ to the line must equal the radius $a$.

So $\frac{|\alpha a - a^2|}{\sqrt{\alpha^2 + \beta^2}} = a \Rightarrow \left(\alpha a - a^2\right)^2 = a^2(\alpha^2 + \beta^2)$.

$\Rightarrow (\alpha - a)^2 = \alpha^2 + \beta^2 \Rightarrow \beta^2 + 2a\alpha = a^2$.

Thus, $(\alpha, \beta)$ lies on the curve $y^2 + 2ax = a^2$.

208. For the circle $x^2 + y^2 = a^2$, the polar of $(x_i, y_i)$ is $xx_i + yy_i = a^2$.

So the polars of $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are

$xx_1 + yy_1 = a^2$, $xx_2 + yy_2 = a^2$, and $xx_3 + yy_3 = a^2$.

These three lines are concurrent if there exists a point $(h, k)$ satisfying all three.

So $hx_1 + ky_1 = a^2$, $hx_2 + ky_2 = a^2$, and $hx_3 + ky_3 = a^2$.

Subtract pairwise: $h(x_1 - x_2) + k(y_1 - y_2) = 0$ and $h(x_2 - x_3) + k(y_2 - y_3) = 0$.

For non-zero $(h, k)$, these two equations imply $(x_1 - x_2)(y_2 - y_3) = (x_2 - x_3)(y_1 - y_2)$.

This is equivalent to $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$.

Thus the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are collinear.

209. The given circles are $x^2 + y^2 - 2x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 6 = 0$.

Compare each with the general form $x^2 + y^2 + 2gx + 2fy + c = 0$.

For the first circle, we get $g_1 = -1$, $f_1 = -3$, and $c_1 = -12$.

For the second circle, we get $g_2 = 3$, $f_2 = 2$, and $c_2 = -6$.

Two circles cut orthogonally if $2(g_1g_2 + f_1f_2) = c_1 + c_2$.

Substituting the values $2((-1)(3) + (-3)(2)) = 2(-3 - 6) = -18$.

Also $c_1 + c_2 = -12 - 6 = -18$.

Both sides are equal. Hence, the given circles cut each other orthogonally.

210. Let the two circles be $S = x^2 + y^2 + 2gx + 2fy + c = 0$ and $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$.

Their radii are $a$ and $a_1$. So $g^2 + f^2 - c = a^2$ and $g_1^2 + f_1^2 - c_1 = a_1^2$.

Consider the circle $\frac{S}{a} + \frac{S_1}{a_1} = 0$.

Its equation is $\left(\frac{1}{a} + \frac{1}{a_1}\right)(x^2 + y^2) + 2\left(\frac{g}{a} + \frac{g_1}{a_1}\right)x + 2\left(\frac{f}{a} + \frac{f_1}{a_1}\right)y + \left(\frac{c}{a} + \frac{c_1}{a_1}\right) = 0$.

$= x^2 + y^2 + 2Gx + 2Fy + C = 0$, where $G = \frac{\frac{g}{a}+\frac{g_1}{a_1}}{\frac{1}{a}+\frac{1}{a_1}}$, $F = \frac{\frac{f}{a}+\frac{f_1}{a_1}}{\frac{1}{a}+\frac{1}{a_1}}$, and $C = \frac{\frac{c}{a}+\frac{c_1}{a_1}}{\frac{1}{a}+\frac{1}{a_1}}$.

Now consider the circle $\frac{S}{a} - \frac{S_1}{a_1} = 0$. Similarly it becomes $x^2 + y^2 + 2G_1x + 2F_1y + C_1 = 0$

where $G_1 = \frac{\frac{g}{a}-\frac{g_1}{a_1}}{\frac{1}{a}-\frac{1}{a_1}}$, $F_1 = \frac{\frac{f}{a}-\frac{f_1}{a_1}}{\frac{1}{a}-\frac{1}{a_1}}$, and $C_1 = \frac{\frac{c}{a}-\frac{c_1}{a_1}}{\frac{1}{a}-\frac{1}{a_1}}$.

Two circles cut orthogonally if $2(GG_1 + FF_1) = C + C_1$. After simplification, both sides reduce to the same value.

Hence, the circles represented by $\frac{S}{a} \pm \frac{S_1}{a_1} = 0$ intersect at right angles.

211. Let the required circles pass through the points $(0, 0)$ and $(0, -a)$.

Then their equation can be taken as $x^2 + y^2 + 2gx + ay = 0$ since substituting $(0, 0)$ and $(0, -a)$ satisfies it.

Now this circle touches the line $y = mx + c$. So the distance of the center $\left(-g, -\frac{a}{2}\right)$ from the line is equal to the radius.

The radius is $\sqrt{g^2 + \frac{a^2}{4}}$. So $\frac{\left|-mg+\frac{a}{2}-c\right|}{\sqrt{m^2+1}} = \sqrt{g^2 + \frac{a^2}{4}}$.

$\Rightarrow \left(-mg + \frac{a}{2} - c\right)^2 = (m^2 + 1)\left(g^2 + \frac{a^2}{4}\right)$. This gives a quadratic in $g$.

The two circles correspond to the two values of $g$. Let them be $g_1$ and $g_2$.

For the two circles to cut orthogonally, the condition is $2\left(g_1g_2 + \frac{a^2}{4}\right) = 0$.

So $g_1g_2 = -\frac{a^2}{4}$.

From the quadratic equation in $g$, the product of roots is $\left(\frac{a}{2} - c\right)^2 - (m^2 + 1)\left(\frac{a^2}{4}\right)$ all divided by $m^2$.

So $g_1g_2 = \frac{\left(\frac{a}{2}-c\right)^2-(m^2+1)\left(\frac{a^2}{4}\right)}{m^2}$. Equate this to $-\frac{a^2}{4}$.

So $\left(\frac{a}{2} - c\right)^2 - (m^2 + 1)\left(\frac{a^2}{4}\right) = -m^2\frac{a^2}{4}$. Solving, $c = \frac{a}{2} \pm \frac{a}{\sqrt{2}}$.

Hence, $c^2 = a^2(2 + m^2)$.

212. Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Given circle one is $x^2 + y^2 + 3x - 5y + 6 = 0$.

So $g_1 = \frac{3}{2}$, $f_1 = -\frac{5}{2}$, $c_1 = 6$. Given circle two is $4x^2 + 4y^2 - 28x + 29 = 0 \Rightarrow x^2 + y^2 - 7x + \frac{29}{4} = 0$.

So $g_2 = -\frac{7}{2}$, $f_2 = 0$, $c_2 = \frac{29}{4}$.

For orthogonality with first circle $2(gg_1 + ff_1) = c + c_1 \Rightarrow 3g - 5f = c + 6$.

For orthogonality with second circle $2(gg_2 + ff_2) = c + c_2 \Rightarrow c = -7g - \frac{29}{4}$.

Substitute in first equation $3g - 5f = -7g - \frac{29}{4} + 6 \Rightarrow f = 2g + \frac{1}{4}$.

Now the center lies on the line $3x + 4y + 1 = 0$. So $3(-g) + 4(-f) + 1 = 0$.

Thus, $3g + 4f = 1$. Substitute $f = 2g + \frac{1}{4} \Rightarrow 3g + 4\left(2g + \frac{1}{4}\right) = 1 \Rightarrow 11g = 0$ so $g = 0$.

Then, $f = \frac{1}{4}$. Now $c = -7g - \frac{29}{4} = -\frac{29}{4}$.

Thus, the required circle is $x^2 + y^2 + \frac{1}{2}y - \frac{29}{4} = 0$.

213. Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since it cuts the circle $x^2 + y^2 = 4$ orthogonally, the condition is $c + (-4) = 0$.

So $c = 4$. Thus the circle becomes $x^2 + y^2 + 2gx + 2fy + 4 = 0$. Its center is $(-g, -f)$.

Given that the center lies on the line $2x - 2y + 9 = 0$, so $2(-g) - 2(-f) + 9 = 0$. Thus, $-2g + 2f + 9 = 0$ or $f = g - \frac{9}{2}$.

Substitute into the equation of the circle $x^2 + y^2 + 2gx + 2\left(g - \frac{9}{2}\right)y + 4 = 0$.

So $x^2 + y^2 + 2g(x + y) - 9y + 4 = 0$. Rewrite as $x^2 + y^2 - 9y + 4 + 2g(x + y) = 0$.

This represents a family of circles depending on $g$.

For fixed points, eliminate $g$. So the condition is $x + y = 0$.

Substitute $y = -x$ into the equation $x^2 + x^2 - 9(-x) + 4 = 0$.

So $2x^2 + 9x + 4 = 0$. $x = -\frac{1}{2}$ or $x = -4$. Thus, the corresponding $y$ values are $\frac{1}{2}$ and 4.

Hence, the two fixed points are $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(-4, 4)$.

214. Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since it cuts the given circles orthogonally,

therefore $2gg_1 + 2ff_2 - c - c_1 = 0$ ...(1) and $2gg_2 + 2ff_2 - c - c_2 = 0$ ...(2)

Eliminating $g$ and $f$ from these equations gives us $\begin{vmatrix} x^2+y^2+c & x & y \\ -c-c_1 & g_1 & f_1 \\ -c-c_2 & g_2 & f_2 \end{vmatrix} = 0$ or

$\begin{vmatrix} x^2+y^2 & x & y \\ -c_1 & g_1 & f_1 \\ -c_2 & g_2 & f_2 \end{vmatrix} + \begin{vmatrix} c & x & y \\ -c & g_1 & f_1 \\ -c & g_2 & f_2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x^2+y^2 & -x & -y \\ c_1 & g_1 & f_1 \\ c_2 & g_2 & f_2 \end{vmatrix} + c\begin{vmatrix} 1 & -x & -y \\ 1 & g_1 & f_1 \\ 1 & g_2 & f_2 \end{vmatrix} = 0$, which is of the form

$$\Rightarrow \begin{vmatrix} x^2+y^2 & -x & -y \\ c_1 & g_1 & f_1 \\ c_2 & g_2 & f_2 \end{vmatrix} + k\begin{vmatrix} -x & -y & 1 \\ g_1 & f_1 & 1 \\ g_2 & f_2 & 1 \end{vmatrix} = 0$$

215. The given circles are $x^2 + y^2 + 5x + 3y + 7 = 0$ and $x^2 + y^2 - 8x + 6y + k = 0$.

Compare them with the general form $x^2 + y^2 + 2gx + 2fy + c = 0$.

For the first circle, $g_1 = \frac{5}{2}$, $f_1 = \frac{3}{2}$, and $c_1 = 7$.

For the second circle, $g_2 = -4$, $f_2 = 3$, and $c_2 = k$.

Two circles cut orthogonally if $2(g_1 g_2 + f_1 f_2) = c_1 + c_2$.

Substitute the values. $2\left(\left(\frac{5}{2}\right)(-4) + \left(\frac{3}{2}\right)(3)\right) = 7 + k$.

Hence, $-11 = 7 + k \Rightarrow k = -18$.

216. Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Since it passes through the origin, $c = 0$.

Now consider the circle $x^2 + y^2 - 4x + 6y + 10 = 0$.

Comparing with the general form, $g_1 = -2$, $f_1 = 3$, and $c_1 = 10$.

Since the circles cut orthogonally, $2(gg_1 + ff_1) = c + c_1$.

Thus, $2(-2g + 3f) = 10 \Rightarrow -2g + 3f = 5$.

Now consider the second circle $x^2 + y^2 + 12y + 6 = 0$.

Comparing with the general form, $g_2 = 0$, $f_2 = 6$, and $c_2 = 6$.

Again using the orthogonality condition, $2(gg_2 + ff_2) = c + c_2 \Rightarrow f = \frac{1}{2}$.

Substitute into $-2g + 3f = 5$. So $-2g + \frac{3}{2} = 5 \Rightarrow g = -\frac{7}{4}$.

Thus, the required circle is $x^2 + y^2 - 7\frac{x}{2} + y = 0$.

217. Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Since it passes through the origin, $c = 0$.

Given that the center lies on the line $x + y + 4 = 0$, so $-g - f + 4 = 0$.

Now consider the circle $x^2 + y^2 - 4x + 2y + 4 = 0$. Comparing with the general form, $g_1 = -2$, $f_1 = 1$, and $c_1 = 4$.

Since the circles cut orthogonally, $2(gg_1 + ff_1) = c + c_1$. So $2(-2g + f) = 4$. Hence, $-2g + f = 2$.

Solving the two equation gives $g = \frac{2}{3}$ and $f = 4 - \frac{2}{3} = \frac{10}{3}$.

Thus, the required circle is $x^2 + y^2 + 4\frac{x}{3} + 20\frac{y}{3} = 0$.

218. Let the third circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Let the other two circles be $x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.

Since the first two circles cut the third circle orthogonally, $2(gg_1 + ff_1) = c + c_1$ and $2(gg_2 + ff_2) = c + c_2$.

The common chord of the first two circles is obtained by subtracting their equations.

So its equation is $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$.

From orthogonality $\Rightarrow c_1 - c_2 = 2g(g_1 - g_2) + 2f(f_1 - f_2)$.

Substitute into the equation of the common chord. Then $2(g_1 - g_2)x + 2(f_1 - f_2)y + 2g(g_1 - g_2) + 2f(f_1 - f_2) = 0$.

Factor, $(g_1 - g_2)(x + g) + (f_1 - f_2)(y + f) = 0$. The center of the third circle is $(-g, -f)$.

Substitute $x = -g$ and $y = -f$. The equation is satisfied.

Hence, the common chord passes through the center of the third circle.

219. Let the required circle be $S = x^2 + y^2 + 2gx + 2fy + c = 0$.

Let $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$, $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and $S_3 = x^2 + y^2 + 2g_3x + 2f_3y + c_3 = 0$.

Since the circle $S = 0$ cuts each of these orthogonally,

$2(gg_1 + ff_1) = c + c_1$, $2(gg_2 + ff_2) = c + c_2$, and $2(gg_3 + ff_3) = c + c_3$.

Now consider the circle $kS_1 + lS_2 + mS_3 = 0$.

Its equation is $(k + l + m)(x^2 + y^2) + 2(kg_1 + lg_2 + mg_3)x + 2(kf_1 + lf_2 + mf_3)y + (kc_1 + lc_2 + mc_3) = 0$.

Divide throughout by $(k + l + m)$. Then the circle becomes $x^2 + y^2 + 2Gx + 2Fy + C = 0$, where $G = \frac{kg_1+lg_2+mg_3}{k+l+m}$, $F = \frac{kf_1+lf_2+mf_3}{k+l+m}$, and $C = \frac{kc_1+lc_2+mc_3}{k+l+m}$.

$2(gG + fF) = \frac{k(2(gg_1+ff_1))+l(2(gg_2+ff_2))+m(2(gg_3+ff_3))}{k+l+m}$.

Using the orthogonality conditions, $2(gG + fF) = \frac{k(c+c_1)+l(c+c_2)+m(c+c_3)}{k+l+m}$.

$2(gG + fF) = c + \frac{kc_1+lc_2+mc_3}{k+l+m} = c + C$.

Hence, the circle $kS_1 + lS_2 + mS_3 = 0$ cuts the circle $S = 0$ orthogonally.

220. Any circle passing through $(0, k)$ and $(0, -k)$ has equation $x^2 + y^2 + 2gx + 2fy + c = 0$.

Since $(0, k)$ lies on it, $k^2 + 2fk + c = 0$. Since $(0, -k)$ lies on it, $k^2 - 2fk + c = 0$.

Subtracting, $4fk = 0$. So $f = 0$. Then $c = -k^2$. Hence, the circle is $x^2 + y^2 + 2gx - k^2 = 0$.

Its center is $(-g, 0)$ and radius is $\sqrt{g^2 + k^2}$.

Now the circle touches the line $y = mx + b$

Therefore, the perpendicular distance of the center from the line equals the radius. So $\frac{|-mg+b|}{\sqrt{m^2+1}} = \sqrt{g^2 + k^2} \Rightarrow b^2 - 2bmg + m^2g^2 = m^2g^2 + g^2 + k^2m^2 + k^2$.

Thus, $g^2 + 2bmg + k^2(m^2 + 1) - b^2 = 0$. This quadratic gives the two possible circles.

Let their corresponding parameters be $g_1$ and $g_2$. Then $g_1g_2 = k^2(m^2 + 1) - b^2$.

Now the two circles are $x^2 + y^2 + 2g_1x - k^2 = 0$ and $x^2 + y^2 + 2g_2x - k^2 = 0$.

They cut orthogonally if $2(g_1 g_2) = -2k^2$.

So $g_1 g_2 = -k^2$. Hence, $k^2(m^2+1) - b^2 = -k^2$. Therefore, $b^2 = k^2(m^2+2)$.

221. Let the required circle be $x^2 + y^2 + 2gx + 2fy + k = 0$.

Since it cuts the circle $x^2 + y^2 = c^2$ orthogonally, the condition is $2(g * 0 + f * 0) = k - c^2$.

So $k = -c^2$. Hence the general equation of the circle is $x^2 + y^2 + 2gx + 2fy - c^2 = 0$.

Now suppose it passes through the point $(a, b)$.

Substituting, $a^2 + b^2 + 2ga + 2fb - c^2 = 0$. So $2ga + 2fb = c^2 - a^2 - b^2$.

Now consider the point $\left(\frac{c^2 a}{a^2+b^2}, \frac{c^2 b}{a^2+b^2}\right)$.

Substitute this point into the equation of the circle. We get $\frac{c^4 a^2}{(a^2+b^2)^2} + \frac{c^4 b^2}{(a^2+b^2)^2} + 2\frac{g(c^2 a)}{a^2+b^2} + 2\frac{f(c^2 b)}{a^2+b^2} - c^2$.

Combine the first two terms, $= c^4 \frac{a^2+b^2}{(a^2+b^2)^2} + \frac{2c^2(ga+fb)}{a^2+b^2} - c^2$.

$= \frac{c^4}{a^2+b^2} + \frac{2c^2(ga+fb)}{a^2+b^2} - c^2$.

Using $2ga + 2fb = c^2 - a^2 - b^2$, we get $2(ga + fb) = c^2 - a^2 - b^2$.

Substitute, $= \frac{c^4}{a^2+b^2} + c^2 \frac{c^2-a^2-b^2}{a^2+b^2} - c^2$.

Simplify, $= \frac{c^4+c^4-c^2(a^2+b^2)}{a^2+b^2} - c^2 = \frac{2c^4-c^2(a^2+b^2)}{a^2+b^2} - c^2 = 0$.

Hence, the circle also passes through $\left(\frac{c^2 a}{a^2+b^2}, \frac{c^2 b}{a^2+b^2}\right)$.

222. Let the given circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be conjugate points with respect to this circle.

Therefore the polar of $P$ passes through $Q$. So $x_1 x_2 + y_1 y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$.

Now consider the circle having $PQ$ as diameter. Its equation is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Comparing with $x^2 + y^2 + 2Gx + 2Fy + C = 0$, we get $G = -\frac{x_1+x_2}{2}$, $F = -\frac{y_1+y_2}{2}$ and $C = x_1 x_2 + y_1 y_2$.

For orthogonality with the given circle, the condition is $2(gG + fF) = c + C$.

Substitute the values, $2(gG + fF) = 2\left(g\left(-\frac{x_1+x_2}{2}\right) + f\left(-\frac{y_1+y_2}{2}\right)\right) = -g(x_1 + x_2) - f(y_1 + y_2)$.

Also $c + C = c + x_1 x_2 + y_1 y_2$.

Using the conjugate point relation, $x_1 x_2 + y_1 y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0$,

we get $c + x_1 x_2 + y_1 y_2 = -g(x_1 + x_2) - f(y_1 + y_2)$.

Hence, $2(gG + fF) = c + C$. Therefore, the circle on $PQ$ as diameter cuts the circle $S$ orthogonally.

223. Equation of common chord of the circle is $x + y - 7 = 0$, which is also the radical axis.

Center of second circle is $(4, 3)$, which lies on the line obtained. Hence, the line is a diamter of the second circle, and hence, the circumference of the second circle is bisected.

224. The given circles are $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ and $x^2 + y^2 + 4x + 2y + 1 = 0$.

Divide the first equation by 2. Then the circles become $S_1 = x^2 + y^2 - x + 3y - \frac{3}{2} = 0$

and $S_2 = x^2 + y^2 + 4x + 2y + 1 = 0$.

Any circle coaxal with them is $S_1 + \lambda S_2 = 0$. So $(1 + \lambda)(x^2 + y^2) + (-1 + 4\lambda)x + (3 + 2\lambda)y + \left(-\frac{3}{2} + \lambda\right) = 0$.

Divide throughout by $(1 + \lambda)$. Then the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where $2g = \frac{-1+4\lambda}{1+\lambda}$ and $2f = \frac{3+2\lambda}{1+\lambda}$.

Hence, the center is $\left(\frac{-4\lambda+1}{2(1+\lambda)}, -\frac{3+2\lambda}{2(1+\lambda)}\right)$.

Subtracting, $5x - y + \frac{5}{2} = 0$. So the center must satisfy $5\left(\frac{-4\lambda+1}{2(1+\lambda)}\right) - \left(-\frac{3+2\lambda}{2(1+\lambda)}\right) + \frac{5}{2} = 0$.

$\Rightarrow 5(-4\lambda + 1) + (3 + 2\lambda) + 5(1 + \lambda) = 0 \Rightarrow -20\lambda + 5 + 3 + 2\lambda + 5 + 5\lambda = 0$.

$\Rightarrow 13 - 13\lambda = 0 \Rightarrow \lambda = 1$.

Therefore the required circle is $S_1 + S_2 = 0$. So $2x^2 + 2y^2 + 3x + 5y - \frac{1}{2} = 0$.

225. The given circles are $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2x^2 + 2y^2 + 3x + 8y + 2c = 0$.

The radical axis is obtained by subtracting the equations. So $\left(2g - \frac{3}{2}\right)x + (2f - 4)y = 0$.

Hence, the radical axis is $(4g - 3)x + (4f - 8)y = 0$.

Now this line touches the circle $x^2 + y^2 + 2x - 2y + 1 = 0$. Its center is $(-1, 1)$ and radius is $\sqrt{1 - 1 + 1} = 1$.

Therefore the perpendicular distance from the center to the line equals the radius. So $\frac{|(4g-3)(-1)+(4f-8)(1)|}{\sqrt{(4g-3)^2+(4f-8)^2}} = 1$.

$\Rightarrow (4g - 3)(f - 2) = 0$. Therefore, either $g = \frac{3}{4}$ or $f = 2$.

226. The given circles are $x^2 + y^2 + 2x + 4y - 6 = 0$ and $x^2 + y^2 = 4$.

Their radical axis is obtained by subtraction. So $\left(x^2 + y^2 + 2x + 4y - 6\right) - \left(x^2 + y^2 - 4\right) = 0$.

Hence, the radical axis is $x + 2y - 1 = 0$.

Let one circle of the required family be $x^2 + y^2 - 4 = 0$.

Then every circle having the same radical axis with it is obtained by adding a multiple of the radical axis.

Hence the required family is $x^2 + y^2 - 4 + \lambda(x + 2y - 1) = 0$.

Therefore the general equation is $x^2 + y^2 + \lambda x + 2\lambda y - (\lambda + 4) = 0$, where $\lambda$ is an arbitrary parameter.

227. Let the required point be $(h, k)$. The square of the length of the tangent from $(h, k)$ to a circle is obtained by substituting the point in the equation of the circle.

For the circle $x^2 + y^2 = 1$, the tangent length squared is $h^2 + k^2 - 1$.

For the circle $x^2 + y^2 - 8x + 15 = 0$, the tangent length squared is $h^2 + k^2 - 8h + 15$.

Since the tangent lengths are equal, $h^2 + k^2 - 1 = h^2 + k^2 - 8h + 15$.

Thus, $h = 2$. Now consider the third circle $x^2 + y^2 + 10y + 24 = 0$.

The tangent length squared is $h^2 + k^2 + 10k + 24$.

Again equating tangent lengths, $h^2 + k^2 - 1 = h^2 + k^2 + 10k + 24 \Rightarrow k = -\frac{5}{2}$.

Therefore, the required point is $\left(2, -\frac{5}{2}\right)$.

228. Let the two circles be $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$.

Let the given point be $P(x_1, y_1)$.

The polar of $P$ with respect to $S_1$ is $xx_1 + yy_1 + g_1(x + x_1) + f_1(y + y_1) + c_1 = 0$.

The polar of $P$ with respect to $S_2$ is $xx_1 + yy_1 + g_2(x + x_1) + f_2(y + y_1) + c_2 = 0$.

These two polars meet at the point $Q$. Subtract the two equations.

Then the coordinates of $Q$ satisfies $(g_1 - g_2)x + (f_1 - f_2)y + g_1x_1 - g_2x_1 + f_1y_1 - f_2y_1 + c_1 - c_2 = 0$.

So $(g_1 - g_2)(x + x_1) + (f_1 - f_2)(y + y_1) + c_1 - c_2 = 0$.

Now the radical axis of the two circles is $2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$.

Let the midpoint of $PQ$ be $(h, k)$. Then $h = \frac{x + x_1}{2}$ and $k = \frac{y + y_1}{2}$.

Substitute in the equation obtained above.

We get $2(g_1 - g_2)h + 2(f_1 - f_2)k + c_1 - c_2 = 0$. But this is exactly the equation of the radical axis. Hence, the midpoint of $PQ$ lies on the radical axis.

229. Let the given points be $A(x_1, y_1)$ and $B(x_2, y_2)$.

Let $P(x, y)$ be a point such that $\frac{PA}{PB} = k$, where $k$ is a constant and $k \neq 1$.

Then $\frac{\sqrt{(x-x_1)^2+(y-y_1)^2}}{\sqrt{(x-x_2)^2+(y-y_2)^2}} = k$.

Squaring, $(x-x_1)^2 + (y-y_1)^2 = k^2\left((x-x_2)^2 + (y-y_2)^2\right)$.

Expand both sides, $x^2 - 2xx_1 + x_1^2 + y^2 - 2yy_1 + y_1^2 = k^2(x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2)$.

$\Rightarrow (1-k^2)(x^2+y^2) + 2(k^2x_2 - x_1)x + 2(k^2y_2 - y_1)y + x_1^2 + y_1^2 - k^2(x_2^2 + y_2^2) = 0$.

Since $k \neq 1$, divide by $(1-k^2)$. The equation becomes of the form $x^2 + y^2 + 2gx + 2fy + c = 0$.

Hence, the locus is a circle.

At $A$, $PA = 0$. So the condition $P\frac{A}{P}B = k$ gives $0 = k$, which is impossible since $k$ is a fixed non-zero constant.

Hence, the circle does not pass through $A$.

Similarly, at $B$, $PB = 0$, so the ratio becomes infinite, which is impossible.

Hence, the circle does not pass through $B$.

230. Let $AB$ and $CD$ be the two rods of lengths $a$ and $b$ respectively.

Let the equation of the circle passing through points $A, B, C$ and $D$ be $x^2 + y^2 - 2\alpha x - 2\beta y + \lambda = 0$, whose center is $P(\alpha, \beta)$.

Putting $y = 0$ gives us $x^2 - 2\alpha x + \lambda = 0 \Rightarrow x_1 + x_2 = 2\alpha, x_1x_2 = \lambda$

$\because a = |x_1 - x_2|, \therefore a^2 = (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = 4(a^2 - \lambda)$

Similarly, $b^2 = 4(b^2 - \lambda) \therefore a^2 - b^2 = 4(\alpha^2 - \beta^2)$.

Hence, the locus of the point $P(\alpha, \beta)$ is $4(x^2 - y^2) = a^2 - b^2$.

231. Let $A$ and $B$ be two fixed points. Let $AB = 2a$, we take the mid-point $O$ of $AB$ as the origin and $OB$ as $x$-axis. Let $A = (-a, 0)$ and $B = (a, 0)$.

Let one straight line which rotates about $B$ makes an angle $\theta$ with the $x$-axis at any time $t$ and at that time the second line which rotates about $A$ makes an angle $2\theta$ with $x$-axis.

Now equations of these lines are $y = \tan\theta(x-a)$ ...(1) and $y = \tan 2\theta(x+a)$

Solving we get $x = \frac{a(\tan 2\theta + \tan\theta)}{\tan\theta - \tan 2\theta}$ and $y = 2a.\frac{\tan 2\theta \tan\theta}{\tan\theta - \tan 2\theta}$

$\Rightarrow x + a = -2a\cos 2\theta$ and $y = -2a\sin 2\theta \Rightarrow (x+a)^2 + y^2 = 4a^2$ is the required locus.

232. Let $OA = a, OB = b$. The equation of the circle through $O, A, B$ is $x^2 + y^2 - ax - by = 0$

Its radius is $r = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(-\frac{b}{2}\right)^2} \Rightarrow a^2 + b^2 = 4r^2$. Here, $a, b$ are variables and $r$ is a constant.

Let $P(\alpha, \beta)$ and $\angle POA = \theta$, then $\alpha = OP\cos\theta = r\cos\theta$ and $\beta = r\sin\theta$

Equation of line $AB$ is $\frac{x}{a} + \frac{y}{b} = 1$. $P$ lies on the line $AB \Rightarrow \frac{\alpha}{a} + \frac{\beta}{b} = 1$.

$\because OP \perp AB \therefore \frac{\beta}{\alpha}\frac{b-0}{0-a} = -1 \Rightarrow a\alpha = b\beta = k(\text{let}) \Rightarrow a = \frac{\alpha}{k}, b = \frac{k}{\beta}$

$\Rightarrow \frac{\alpha}{\frac{k}{\alpha}} + \frac{\beta}{\frac{k}{\beta}} = 1 \Rightarrow \alpha^2 + \beta^2 = k \Rightarrow a = \frac{\alpha^2+\beta^2}{\alpha}, b = \frac{\alpha^2+\beta^2}{\beta}$

$\Rightarrow \left(\alpha^2 + \beta^2\right)^2\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right) = 4r^2$

Thus, locus of point $P$ is $\left(x^2 + y^2\right)^2\left(\frac{1}{x^2} + \frac{1}{y^2}\right) = 4r^2$.

233. Let $P(\alpha, \beta)$ be the point whose locus is to be found. Let the given circle be $x^2 + y^2 = a^2$, and tangent to this circle is $y = mx + a\sqrt{1+m^2}$ which passes through $P$.

Thus, $\beta = m\alpha + a\sqrt{1+m^2} \Rightarrow m^2(a^2 - \alpha^2) + 2m\alpha\beta + a^2 - \beta^2 = 0$.

This is a quadratic equation in $m$ and hance two values are possible. Thus, these lines will be orthogonal if $m_1m_2 = -1 \Rightarrow \frac{a^2-\beta^2}{a^2-\alpha^2} = -1 \Rightarrow \alpha^2 + \beta^2 = 2a^2$.

Thus, locus of $P(\alpha, \beta)$ is $x^2 + y^2 = 2a^2$.

234. Given is the parametric equation of the circle. The cartesian equation will be $x^2 + y^2 = a^2$.

Let $A = (a\cos\theta, a\sin\theta), B = \left(a\cos\left(\theta + \frac{\pi}{3}\right), a\sin\left(\theta + \frac{\pi}{3}\right)\right)$.

Equation of tangent at $A$ is $xa\cos\theta + ya\sin\theta = a^2 \Rightarrow x\cos\theta + y\sin\theta = a$.

Similarly, equation of tangent at $B$ is $x\cos\left(\theta + \frac{\pi}{3}\right) + y\sin\left(\theta + \frac{\pi}{3}\right) = a$

$\Rightarrow x\cos\theta.\frac{1}{2} - x\sin\theta\frac{\sqrt{3}}{2} + y\sin\theta.\frac{1}{2} + y\cos\theta\frac{\sqrt{3}}{2} = a$

$\Rightarrow \frac{1}{2}(x\cos\theta + y\sin\theta) + \frac{\sqrt{3}}{2}(y\cos\theta - x\sin\theta) = a$

$\Rightarrow \frac{a}{2} + \frac{\sqrt{3}}{2}(y\cos\theta - x\sin\theta) = a \Rightarrow y\cos\theta - x\sin\theta = \frac{a}{\sqrt{3}}$

Squaring and adding with the equation of tangent at $A$ yields

$3(x^2 + y^2) = 4a^2$, which is the required locus.

235. Equation of chord of intersection is $2(a - b)x = 0 \Rightarrow x = 0$.

Thus, $x = 0$ is the equation of the chord of intersection. $OA^2 = \sqrt{a^2 + c^2 - a^2} = a$.

Since common chord of the two circles is $y$-axis and their centers are $(-a, 0)$ and $(-b, 0)$ lying on the $x$-axis.

Therefore, one of $a$ and $b$ will be positive and other negative. WLOG we can assume that $a < 0, b > 0$ with $|a| < b$.

Let $AP$ be an arbitrary line through $A(0, c)$ which meets first circle at $P(x_2, y_2)$. Let the slope of $AP$ be $m$.

Equation of $AP$ is $y = mx + c$ and that of $BQ$ is $y = mx - c$.

Let $Q = (x_3, y_3)$. Let $R(\alpha, \beta)$ be the mid-point of $PQ$. Putting $y = mx + c$ in first circle yields

$x^2+(mx+c)^2+2ax-c^2=0 \Rightarrow x=0, -\left(2\frac{a+cm}{1+m^2}\right)$

$\because x_2 \neq 0 \therefore x_2 = -\left(2\frac{a+cm}{1+m^2}\right)$ and $y_2 = -\left(2\frac{m(a+cm)}{1+m^2}\right)+c$

Replacing $a$ by $b$ and $c$ by $-c$ gives us

$x_3 = \left(2\frac{b-cm}{1+m^2}\right)$ and $y_3 = -\left(2\frac{m(b-cm)}{1+m^2}\right)-c$

$\alpha = \frac{x_2+x_3}{2} = -\frac{a+b}{1+m^2}, \beta = -\frac{m(a+b)}{1+m^2} \Rightarrow \frac{\beta}{\alpha} = m.$

$\Rightarrow \alpha = -\frac{(a+b)\alpha^2}{\alpha^2+\beta^2} \Rightarrow \alpha^2+\beta^2+(a+b)\alpha = 0$

Hence, the locus of $(\alpha,\beta)$ is $x^2+y^2+(a+b)x=0$.

236. The chord of contact of tangents drawn from $(\alpha,\beta)$ to the circle $x^2+y^2=a^2$ is $\alpha x+\beta y=a^2$.

This chord subtends a right angle at the center $(0,0)$. Let the perpendicular distance from the center to the chord be $d$.

For a chord of a circle of radius $a$ subtending a right angle at the center, $d = a\cos(\frac{\pi}{4}) = \frac{a}{\sqrt{2}}$.

Now the distance of the center from the chord $\alpha x+\beta y-a^2=0$ is $\frac{a^2}{\sqrt{\alpha^2+\beta^2}}$.

Therefore, $\frac{a^2}{\sqrt{\alpha^2+\beta^2}} = \frac{a}{\sqrt{2}}$. So $a^4 = a^2\frac{\alpha^2+\beta^2}{2}$.

Hence, $\alpha^2+\beta^2=2a^2$. Therefore, the required condition is $\alpha^2+\beta^2=2a^2$.

Thus, the locus of $(\alpha,\beta)$ is the circle $x^2+y^2=2a^2$.

237. Let the tangents to the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$ intersect at the point $(h,k)$.

The tangent from $(h,k)$ to the first circle has equation $y=mx\pm\sqrt{a^2(1+m^2)}$.

Since it passes through $(h,k)$, $k=mh\pm\sqrt{a^2(1+m^2)}$. $\Rightarrow (k-mh)^2=a^2(1+m^2)$.

This gives the combined equation of tangents from $(h,k)$ to the circle $x^2+y^2=a^2$,

namely $\left(xh+yk-a^2\right)^2 = \left(h^2+k^2-a^2\right)\left(x^2+y^2-a^2\right)$.

Similarly, the pair of tangents from $(h,k)$ to the circle $x^2+y^2=b^2$ is

$\left(xh+yk-b^2\right)^2 = \left(h^2+k^2-b^2\right)\left(x^2+y^2-b^2\right)$.

Now the two tangents are mutually perpendicular. For a pair of tangents drawn from a point to the circle $x^2+y^2=r^2$, the angle between them is a right angle if $h^2+k^2=2r^2$.

Applying this separately to the two circles and combining for perpendicular tangents, we obtain $h^2+k^2=a^2+b^2$.

Hence, the locus of the point of intersection is $x^2+y^2=a^2+b^2$. This is a circle concentric with the given circles.

238. Let $(h, k)$ be the point from which tangents are drawn to the circle $x^2 + y^2 = a^2$.

Let the angle between the tangents be $\alpha$. If $P$ is the external point and $O$ is the center, then in the right triangle formed by joining the center to the point of contact,

$\sin\left(\frac{\alpha}{2}\right) = \frac{a}{\sqrt{h^2+k^2}}$. Hence, $h^2 + k^2 = a^2 \csc^2\left(\frac{\alpha}{2}\right)$.

Now use the identity $\csc^2\left(\frac{\alpha}{2}\right) = \frac{\tan^2 \alpha+4}{4\tan^2\left(\frac{\alpha}{2}\right)}$.

After simplification, $\left(h^2 + k^2 - 2a^2\right)^2 \tan^2 \alpha = 4a^2\left(h^2 + k^2 - a^2\right)$.

Replacing $(h, k)$ by the general point $(x, y)$, the locus is $\left(x^2 + y^2 - 2a^2\right)^2 \tan^2 \alpha = 4a^2\left(x^2 + y^2 - a^2\right)$.

239. Let the variable line through the fixed point $(h, k)$ have slope $m$. Its equation is $y - k = m(x - h)$.

So $mx - y + (k - mh) = 0$. Let $(x, y)$ be the foot of the perpendicular drawn from the origin to this line.

Since $(x, y)$ lies on the line, $mx - y + k - mh = 0$.

Also the line joining the origin to $(x, y)$ is perpendicular to the given line.

The slope of the given line is $m$. Hence, the slope of the perpendicular from the origin is $-\frac{1}{m}$.

Therefore, $\frac{y}{x} = -\frac{1}{m}$, so $m = -\frac{x}{y}$.

Substitute this in the line equation, $\left(-\frac{x}{y}\right)x - y + k - \left(-\frac{x}{y}\right)h = 0 \Rightarrow -x^2 - y^2 + ky + hx = 0$.

Hence, the locus is $x^2 + y^2 - hx - ky = 0$.

240. Take the fixed point $O$ as the origin. Let the two fixed parallel lines be $x = a$ and $x = -b$.

Then the points $A(a, 0)$ and $B(-b, 0)$ lie on the perpendicular through $O$.

Let $P(a, p)$ and $Q(-b, q)$. Since $\angle POQ$ is a right angle, the slopes of $OP$ and $OQ$ satisfy

$\left(\frac{p}{a}\right)\left(\frac{q}{-b}\right) = -1$. So $pq = ab$. Now find the equation of the line $PQ$.

Using the two-point form, $y - p = (q - p)\frac{x-a}{-b-a}$.

This simplifies to $(p - q)x + (a + b)y - (aq + bp) = 0$.

Let $(h, k)$ be the foot of the perpendicular from the origin to this line.

Then $h = -(p - q)\frac{-aq-bp}{(p-q)^2+(a+b)^2}$ and $k = -(a + b)\frac{-aq-bp}{(p-q)^2+(a+b)^2}$.

So $h = (p - q)\frac{aq+bp}{(p-q)^2+(a+b)^2}$ and $k = (a + b)\frac{aq+bp}{(p-q)^2+(a+b)^2}$.

Using $pq = ab$, simplification gives $h^2 + \left(k^2\right) = (a - b)h$.

Rewrite, $h^2 - (a - b)h + k^2 = 0$. Complete the square, $\left(h - \frac{a-b}{2}\right)^2 + k^2 = \left(\frac{a+b}{2}\right)^2$.

This is the equation of the circle whose diameter has endpoints $(a, 0)$ and $(-b, 0)$, that is, the circle on $AB$ as diameter.

Hence, the locus of the foot of the perpendicular from $O$ to $PQ$ is the circle on $AB$ as diameter.

241. Let the other end of the diameter through $P(1, 2)$ be $(x, y)$. Let the center of the circle be $(h, k)$.

Since the center is the midpoint of the diameter joining $(1, 2)$ and $(x, y)$, $h = \frac{x+1}{2}$ and $k = \frac{y+2}{2}$.

The circle touches the $x$-axis. Therefore, the radius equals the distance of the center from the $x$-axis.

So the radius is $k$. Now the radius is also half the length of the diameter.

Hence, $\frac{(x-1)^2+(y-2)^2}{4} = k^2$. Substitute $k = \frac{y+2}{2}$.

Then $\frac{(x-1)^2+(y-2)^2}{4} = \frac{(y+2)^2}{4} \Rightarrow (x-1)^2 = 8y$.

242. Let the required point be $(x, y)$. The length of the tangent from $(x, y)$ to the circle $x^2 + y^2 = a^2$ is $\sqrt{x^2 + y^2 - a^2}$.

Similarly, the length of the tangent from $(x, y)$ to the circle $x^2 + y^2 = b^2$ is $\sqrt{x^2 + y^2 - b^2}$.

Given that the tangent lengths vary inversely as the radii, $\frac{\sqrt{x^2+y^2-a^2}}{\sqrt{x^2+y^2-b^2}} = \frac{b}{a}$.

$\Rightarrow \frac{x^2+y^2-a^2}{x^2+y^2-b^2} = \frac{b^2}{a^2} \Rightarrow a^2(x^2 + y^2 - a^2) = b^2(x^2 + y^2 - b^2)$.

Hence, $x^2 + y^2 = a^2 + b^2$. Therefore, the locus is the circle $x^2 + y^2 = a^2 + b^2$.

243. Take the square with sides parallel to the axes and center at the origin.

Since the side of the square is unity, its sides are $x = \frac{1}{2}$, $x = -\frac{1}{2}$, $y = \frac{1}{2}$, and $y = -\frac{1}{2}$.

Let $(x, y)$ be the moving point. Its perpendicular distances from the four sides are $|x - \frac{1}{2}|$, $|x + \frac{1}{2}|$, $|y - \frac{1}{2}|$, and $|y + \frac{1}{2}|$.

Given that the sum of their squares is 9, $\left(x - \frac{1}{2}\right)^2 + \left(x + \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = 9$.

$\Rightarrow x^2 - x + \frac{1}{4} + x^2 + x + \frac{1}{4} + y^2 - y + \frac{1}{4} + y^2 + y + \frac{1}{4} = 9$.

$\Rightarrow x^2 + y^2 = 4$. This is a circle centered at the origin, which is the center of the square.

Therefore the locus is a circle concentric with the square. Its radius is 2.

244. The given circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Its center is $(-g, -f)$.

Let the center be $(h, k)$. Then $g = -h$ and $f = -k$.

The length of the tangent from the origin to the circle is $\sqrt{c}$ evaluated at the origin, so $\sqrt{0 + 0 + 0 + 0 + c} = \sqrt{c}$.

The pair of tangents drawn from the origin are perpendicular.

For tangents from a point to a circle to be perpendicular, the point must lie on the director circle.

The director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$.

Since the origin lies on it, $2c - g^2 - f^2 = 0$. So $g^2 + f^2 = 2c$.

Now the radius squared of the circle is $g^2 + f^2 - c$. Substitute the above relation, $\text{radius}^2 = c$.

Thus, $g^2 + f^2 = 2c$. Replacing $g = -h$ and $f = -k$, $h^2 + k^2 = 2c$.

Hence, the locus of the center is $x^2 + y^2 = 2c$.

245. Let the required circle have center $(h, k)$ and radius $r$. The circle $x^2 + y^2 = a^2$ has center $(0, 0)$ and radius $a$.

The circle $x^2 + y^2 = 4ax$. So its center is $(2a, 0)$ and radius is $2a$.

Since the required circle touches both circles externally, $\sqrt{h^2 + k^2} = r + a$, and $\sqrt{(h - 2a)^2 + k^2} = r + 2a$.

Subtracting, $\sqrt{(h - 2a)^2 + k^2} - \sqrt{h^2 + k^2} = a$.

Let $d_1 = \sqrt{h^2 + k^2}$, and $d_2 = \sqrt{(h - 2a)^2 + k^2}$.

Then $d_2 = d_1 + a$. Squaring, $(h - 2a)^2 + k^2 = h^2 + k^2 + 2ad_1 + a^2$.

Simplify, $h^2 - 4ah + 4a^2 + k^2 = h^2 + k^2 + 2ad_1 + a^2$.

So $-4ah + 3a^2 = 2ad_1$. Hence, $d_1 = \frac{3a-4h}{2}$.

Now square again, $h^2 + k^2 = \frac{(3a-4h)^2}{4}$. Thus, $4h^2 + 4k^2 = 9a^2 - 24ah + 16h^2$.

Therefore, $12h^2 - 4k^2 - 24ah + 9a^2 = 0$.

Replacing $(h, k)$ by $(x, y)$, the locus is $12x^2 - 4y^2 - 24ax + 9a^2 = 0$.

246. Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. Its center is $(-g, -f)$.

The given circles are $x^2 + y^2 + 4x - 6y + 9 = 0$, and $x^2 + y^2 - 4x + 6y + 4 = 0$.

For the first circle, $g_1 = 2$, $f_1 = -3$, and $c_1 = 9$.

Since the required circle cuts it orthogonally, $2(gg_1 + ff_1) = c + c_1$.

Thus, $4g - 6f = c + 9$. For the second circle, $g_2 = -2$, $f_2 = 3$, and $c_2 = 4$.

Again using orthogonality, $2(gg_2 + ff_2) = c + c_2$.

So $-4g + 6f = c + 4$. Subtract the two equations, $(4g - 6f) - (-4g + 6f) = (c + 9) - (c + 4)$.

Hence $8g - 12f = 5$. Now let the center be $(x, y)$. Then $g = -x$ and $f = -y$.

Substitute, $8(-x) - 12(-y) = 5$. So $-8x + 12y = 5$. Therefore, $8x - 12y + 5 = 0$.

247. Let the fixed point on the $x$-axis be $(c, 0)$. Take any tangent to the circle $x^2 + y^2 = a^2$.

Its equation may be written as $y = mx \pm a\sqrt{1+m^2} \Rightarrow mx - y \pm a\sqrt{1+m^2} = 0$.

Let $(h, k)$ be the foot of the perpendicular from $(c, 0)$ to this tangent. Since $(h, k)$ lies on the tangent,

$mh - k \pm a\sqrt{1+m^2} = 0$.

Also the line joining $(c, 0)$ to $(h, k)$ is perpendicular to the tangent.

Hence its slope is $-\frac{1}{m}$. Therefore, $\frac{k}{h-c} = -\frac{1}{m}$, so $m = -\frac{h-c}{k}$.

Substitute in the tangent equation, $-\frac{h(h-c)}{k} - k \pm a\sqrt{1 + \frac{(h-c)^2}{k^2}} = 0$.

$\Rightarrow -h(h-c) - k^2 \pm a\sqrt{k^2 + (h-c)^2} = 0$.

Transpose, $a\sqrt{(h-c)^2 + k^2} = h(h-c) + k^2$.

Now square and simplify. After reduction, $\left(h^2 + k^2 - ch\right)^2 = a^2\left((h-c)^2 + k^2\right)$.

Replacing $(h, k)$ by $(x, y)$, the locus is $\left(x^2 + y^2 - cx\right)^2 = a^2\left((x-c)^2 + y^2\right)$.

248. Let the point $P$ on the circle $x^2 + y^2 = 2$ be $(x_1, y_1)$. Then $x_1^2 + y_1^2 = 2$.

The tangent at $P$ is $xx_1 + yy_1 = 2$. This tangent cuts the $x$-axis at the point $L$.

Putting $y = 0$, $xx_1 = 2$, so $L\left(\frac{2}{x_1}, 0\right)$.

Similarly, it cuts the $y$-axis at the point $M$. Putting $x = 0$, $yy_1 = 2$, so $M\left(0, \frac{2}{y_1}\right)$.

Let the midpoint of $LM$ be $(h, k)$. Then $h = \frac{1}{x_1}$, and $k = \frac{1}{y_1}$.

Therefore, $x_1 = \frac{1}{h}$ and $y_1 = \frac{1}{k}$. Substitute in $x_1^2 + y_1^2 = 2$.

We get $\frac{1}{h^2} + \frac{1}{k^2} = 2$. Hence, the locus is $x^2 + y^2 = 2x^2y^2$.

249. Let the triangle have vertices $(0, 0)$, $(h, 0)$ and $(0, k)$. Its third side joins $(h, 0)$ and $(0, k)$.

Hence its equation is $\frac{x}{h} + \frac{y}{k} = 1$. Or, $kx + hy - hk = 0$.

This line touches the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$.

So the center is $(a, a)$ and the radius is $a$.

Therefore the perpendicular distance from $(a, a)$ to the line equals $a$.

Thus, $|ak + ah - hk\frac{|}{\sqrt{h^2+k^2}} = a$.

Squaring, $hk\left(hk - 2ah - 2ak + 2a^2\right) = 0 \Rightarrow hk - 2ah - 2ak + 2a^2 = 0$.

Now the triangle is right-angled at the origin, so the circumcenter is the midpoint of the hypotenuse.

Hence, the circumcenter is $\left(\frac{h}{2}, \frac{k}{2}\right)$. Let it be $(x, y)$. Then $h = 2x$, and $k = 2y$.

Substitute, $(2x)(2y) - 2a(2x) - 2a(2y) + 2a^2 = 0$. So $4xy - 4ax - 4ay + 2a^2 = 0$.

$\Rightarrow 2(x+y)-a=2x\frac{y}{a}$.

250. Let the moving point on the circle $x^2+y^2=4$ be $(x_1, y_1)$.

Let the midpoint of $AP$ be $(h, k)$. Since $A=(1,5)$ and $P=(x_1, y_1)$, the midpoint is $h=\frac{x_1+1}{2}$ and $k=\frac{y_1+5}{2}$.

Hence, $x_1=2h-1$ and $y_1=2k-5$. Since $P$ lies on the circle, $x_1^2+y_1^2=4$.

Substitute, $(2h-1)^2+(2k-5)^2=4 \Rightarrow 2h^2+2k^2-2h-10k+11=0$.

Replacing $(h, k)$ by $(x, y)$, the locus is $2x^2+2y^2-2x-10y+11=0$.

251. Let $P(x, y)$ be the midpoint of a variable chord through the fixed point $A(a, b)$ of the circle $x^2+y^2+2gx+2fy+c=0$.

Let the other end of the chord be $Q(x_1, y_1)$. Since $P$ is the midpoint of $AQ$, $x_1=2x-a$ and $y_1=2y-b$.

Now $Q$ lies on the circle. Therefore, $(2x-a)^2+(2y-b)^2+2g(2x-a)+2f(2y-b)+c=0$.

$\Rightarrow 4x^2-4ax+a^2+4y^2-4by+b^2+4gx-2ag+4fy-2bf+c=0$.

$\Rightarrow 4(x^2+y^2+gx+fy)-4(ax+by)+a^2+b^2-2ag-2bf+c=0$.

$\Rightarrow x^2+y^2+gx+fy-ax-by+\frac{a^2+b^2-2ag-2bf+c}{4}=0$.

This is a circle. The center is $\left(\frac{a-g}{2}, \frac{b-f}{2}\right)$. The center of the given circle is $(-g,-f)$.

The midpoint of the points $(a, b)$ and $(-g,-f)$ is $\left(\frac{a-g}{2}, \frac{b-f}{2}\right)$.

Hence the locus is a circle whose center is the midpoint of the fixed point $A$ and the center of the given circle.

Also its radius is half the radius of the given circle.

Therefore, the locus is the circle obtained by reducing the given circle in the ratio $1:2$ with respect to the point $A$.

252. Let the two fixed points be $A(x_1, y_1)$ and $B(x_2, y_2)$. Let the moving line be $lx+my+n=0$, where $l^2+m^2=1$.

Then the algebraic perpendicular distances of the points from the line are $lx_1+my_1+n$ and $lx_2+my_2+n$.

Given that their algebraic sum is constant, say $2k$, $(lx_1+my_1+n)+(lx_2+my_2+n)=2k$.

So $l(x_1+x_2)+m(y_1+y_2)+2n=2k$. Hence, $n=k-\frac{l(x_1+x_2)+m(y_1+y_2)}{2}$.

Substitute this in the equation of the line, $lx+my+k-\frac{l(x_1+x_2)+m(y_1+y_2)}{2}=0$.

Rearrange, $l\left(x-\frac{x_1+x_2}{2}\right)+m\left(y-\frac{y_1+y_2}{2}\right)+k=0$.

Since $l^2+m^2=1$, this represents the tangent form of a circle.

Therefore the line always touches the fixed circle whose center is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

and radius is $|k|$.

Hence, the required fixed circle is $\left(x - \frac{x_1+x_2}{2}\right)^2 + \left(y - \frac{y_1+y_2}{2}\right)^2 = k^2$.

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Version 1.3, 3 November 2008



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